Decodability Analysis of Finite Memory Random Linear Coding in Line Networks

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***Abstract*—We consider the problem of decodability when random linear coding (RLC) is performed on a stream of packets in a line network. First, we clearly define the problem of decodability for a stream of arriving packets, and discuss its importance with some examples. Then, we will find the limits on the mean arrival rate under which the stream is decodable. Further, upper bounds will be derived for the average length of a decoded block of packets in multi-hop line networks. Finally, these analytical results are validated via simulations.**

I. Introduction

Linear network codes achieve the min-cut capacity of networks [1]. In fact, random linear codes over large Galois fields have been proved to do so [2]. In [3], a distributed network coding scheme was introduced, where each node stores the arriving packets and forwards random linear combinations of the stored packets. As a result, for a network with no buffer limitations, all arriving packets are stored, and then used to generate new packets. Hence, there is no information loss. However, in this case, upon reception of a packet, a node has to determine whether or not the incoming packet is in the linear span of its previously stored packets. Further, for generating every coded packet, all stored packets need to be accessed. It is therefore desirable to have limited buffer sizes, since it limits the complexity of storage and coded packet generation process. Further, using small buffers at relay nodes simplifies practical issues such as on-chip board space and memory-access latency as well as reducing the average packet delay [4].

Various coding strategies for achieving capacity in infinite-buffer erasure line networks is outlined in [5]. Later, [6] considered the limitations posed by finite memory, specifically in a simple line network involving a single intermediate node. Several challenges arise when extending the study from a single intermediate node to a multi-hop line network as detailed in [7]. However, none of the previous works in the literature have addressed the problem of decodability for finite-memory random linear network coding.

In this paper, our objective is to define the problem of decodability when random linear network coding is performed on a stream of arriving packets and also to derive bounds on decodable rates and decoded block lengths in line networks.

II. Notations and Definitions

We consider a memoryless packet arrival process with mean rate λ for the source which is able to accommodate infinitely many packets until they are decoded at the destination. The block of packets that are decoded will then be deleted from the source buffer. We consider a line network of hop-length h, a graph with vertex set $V = \{s = v_0, v_1, v_2, ..., v_{h-1}, d = v_h\}$ and edge set $\overrightarrow{e} = \{\{v_i, v_{i+1}\} : i = 0, ..., h-1\}$ with erasure probability ε_i on link $\{v_{i-1}, v_i\}$ for $i = 1, ..., h$. It is assumed that random linear coding (RLC) over $\mathbb{F}_q$ is performed at the source as well as the intermediate nodes, where $\mathbb{F}_q$ is the Galois field of size q. [1] Moreover, we employ the following notations. For any $x \in [0, 1]$, $\overline{x} \triangleq 1 - x$. Node s and node d represent the source and destination nodes, respectively.

III. Maximum Decodable Throughput

To define the problem of decodability, first we have to identify the rules and conditions under which a block of RLC-encoded packets is decoded at the destination. As an example, similar to the model in [6], assume the source (encoder) has a finite memory of size m. Further, the destination (decoder) receives packets directly from the source, *i.e.*, there is no relay node. For now, we define the state of the network as the difference between the number of packets arrived at the source and the number of packets received by the destination, *i.e.*, transmitted to the destination and not lost. At the beginning of the first time epoch, the memory of the source is empty and we are in state 0. We remain in this state until the first packet p_1 arrives. Suppose the next packet transmitted from the source to the destination is not lost. Then we still remain in state 0, but the destination receives a packet that is a random linear combination of only the packet p_1, *i.e.*, a random scalar multiple of p_1. Hence, the decoder recovers p_1 from the received packet. Now suppose after the first packet p_1 arrives, the next outgoing packet is lost and we reach state 1. Suppose packet p_2 arrives before an outgoing packet is successfully transmitted, *i.e.*, transmitted and not lost. Then, any packet to be transmitted by the source is a random linear combination of p_1 and p_2. Suppose further that a packet is received by the destination, so we are again in state 1. This packet is currently useless to the destination node, since it is neither p_1 nor p_2. Nevertheless, it contains some information previously unknown to the destination node, *e.g.* p_1 and p_2 lie in a certain linear subspace. Consequently, the next packet received by the destination delivers previously unknown information, provided that it is linearly independent of the packet already stored.

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[1] Throughout this work, we only consider the case where q is sufficiently large.

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Such a packet is called an "innovative" packet. Further, it is notable that packets p_1 and p_2 will be decoded simultaneously at the destination and hence will generate a *decoded block* of length 2. Basically, every packet that is transmitted from a non-zero state is innovative at the destination because we assume q is sufficiently large. Also, every time the state returns to 0, a block of packets will be decoded and the length of the decoded block corresponds to the number of packets arrived during the time that the state was non-zero.

However, as claimed in [6], the statement above is true only when packets arrive at the source in states $0, 1, \ldots, m-1$. If a packet arrives at source in state m, the current contents of the source will be overwritten and hence corrupted, and will never be recovered. This is because a source with a buffer size m can only generate m innovative packets and after that any linear combination would be linearly dependent to the previously generated ones. In other words, the source has exactly m innovative packets to transmit before and after receiving the new packet, meaning that a packet worth of information is lost by this arrival. Moreover, the current contents of the source are corrupted and impossible to recover. In [6], the probability of packet loss is defined to characterize such behavior. In this work, however, an infinitely large buffer size is assumed for the source to investigate the characteristics and behavior of the decoding process at the destination for a multi-hop line network, without having to worry about packet loss or corruption of the contents of the source buffer. We will realize that such advantages come at the cost of decoding delay, *i.e.*, occasionally having to wait a long time for a block of packets to be decoded. We call a stream of packets with a fixed mean arrival rate λ *decodable* if the expected waiting time for a block of packets to be decoded is finite. The mean arrival rate λ associated with a decodable stream will be called a *decodable arrival rate*. However, the question is whether there are any arrival rates for which a stream of packets is not decodable. To answer this question, next, we will define parameters that have a critical role in characterizing decodability.

In the example above, we realize that each coded packet received at the destination is in fact a linear "equation" for which the original information packets arrived at the source are its "unknowns" to be found. Hence, upon receiving as many linearly independent equations at the destination as the number of unknowns, the system of linear equations is solvable and hence, a block of packets is decoded. The size of the decoded blocks is equal to the number of unknowns at the moment the system of linear equations is solved. Therefore, to guarantee that a stream of packets with arrival rate λ is decodable, the number of unknowns received at the destination should not grow unboundedly with respect to the number of equations received. To address such a problem, we need to be able to characterize the growth rate and dependencies of both the number of innovative packets at the destination (equations) and the number of original packets used in those innovative packets (unknowns). Previously, in [7], the authors have developed analytical results regarding the arrival rate of innovative packets at the destination when the network performs at steady state, *i.e.*, throughput.

In a line network setting, we define the innovativeness of node v_i with respect to node v_{i+1} at time epoch t, denoted by $I_i(t)$, as the number of packets stored in v_i that are innovative for v_{i+1}. The innovativeness of a node is limited to its buffer size, *i.e.*, $0 \leq I_i(t) \leq m_{v_i}$. Further, each arrival at the source increases its innovativeness, $I_S(t)$, by one. With RLC being performed on potentially a large number of information packets at the source, the buffer of the intermediate nodes contains a limited number of linearly independent packets (equations) including a large number of source-originated Packets (unknown variables). For the purpose of decoding analysis, in addition to the innovativeness of each node, the number of original packets involved in the buffer contents of each intermediate node is also considered. Hence, we define $P_i(t)$ as the number of original packets used in forming the linear combinations stored at the buffer of node v_i.

A. Decodability condition for a Two-hop Line Network

In this section, for simplicity of representation, we consider three nodes: A source S, a relay R, and a destination D. Further, their innovativeness are denoted by $I_S(t)$, $I_R(t)$, $I_D(t)$, and the number of original packets used in forming the linear combinations stored at their buffers are denoted by $P_S(t)$, $P_R(t)$, $P_D(t)$, respectively.

Previously, we have seen how the innovativeness of each node changes with arrival and/or departure of packets. For example, I_S increases by one with each packet arrival at the source, but potentially[2] decreases by one when a packet is transmitted successfully while $I_R < m$, where m is the buffer size of the relay R. Further, I_R potentially[3] increases by one if a packet is successfully transmitted to the relay from the source when $I_S > 0$, and potentially decreases by one when a packet is transmitted successfully to the destination. Finally, I_D only increases by one if a packet is successfully transmitted to the destination from the relay node when $I_R > 0$.

The changes in parameters $P_S(t)$, $P_R(t)$, $P_D(t)$ are quite different from how innovativeness of each node behaves. For the source node, since each arriving packet contributes a new unknown variable for decoding, $P_S(t)$ increases by one with each packet arrival at the source. Further, $P_R(t)$ either remains the same or takes the value of $P_S(t-1)$ where the latter occurs when a packet is received at relay R from the source no matter what are the buffer contents. In other words, when a packet is transmitted by the source and not lost, it brings a linear combination of all the packets stored at the source and combines it with the previously stored contents of the relay. Similarly, $P_D(t)$ either remains the same or takes the value of $P_R(t-1)$ where the latter occurs when a packet is received at the destination. Note that, the above changes occur regardless of the innovativeness of the packets. To summarize, let $B_p(t)$ be a Bernoulli random variable taking the value 1 with probability p at time epoch t and the value 0 otherwise. The following represents the changes in $P_S(t)$, $P_R(t)$, $P_D(t)$ in terms of λ, ε_1, and ε_2.

$$
\begin{aligned}
P_S(t+1) &= P_S(t) + B_\lambda(t) && (1)\\
P_R(t+1) &= P_R(t) + B_{\overline{\varepsilon}_1}(t)\left(P_S(t) - P_R(t)\right) && (2)\\
P_D(t+1) &= P_D(t) + B_{\overline{\varepsilon}_2}(t)\left(P_R(t) - P_D(t)\right) && (3)
\end{aligned}
$$

[2] I_S is non-negative.

[3] I_R should not exceed m.

The following lemma summarizes the necessary and sufficient conditions for a block of packets to be decoded using the parameters described above.

Lemma 1 *A block of length K is decoded at time t^* if and only if both the following relations hold:*

1) $P_D(t^*) = I_D(t^*)$
2) $I_D(t^*) - I_D(t_0) = K$, *where* $t_0 = \max\left(\{t < t^* : P_D(t) = I_D(t)\}\right)$

The proof can be found in Appendix A.

Lemma 1 only presents the conditions for a single event of decoding of a block of packets. However, we are more interested in conditions that must hold to ensure the decodability of a stream of packets in the long run. At the beginning of this section, using a toy example, we observed that every time the state of the source returns to 0, a block of packets will be decoded. Although this statement is not true for a general multi-hop line network, later we will see that at steady-state, a block of packets is decoded if and only if the source revisits the state 0 at least once before the moment of decoding. Lemma 3 will present the necessary and sufficient condition for decodability at steady-state.

Lemma 2 *The ordered tuple $(I_S(t), I_R(t))$ forms an irreducible Markov chain.*

The proof can be found in Appendix B.

Lemma 3 *A stream of packets with source arrival rate λ is decodable if and only if in the Markov chain $(I_S(t), I_R(t))$, any state of the form $(0, Y)$ is recurrent, where $Y = 0, 1, \ldots, m$.*

The proof can be found in Appendix C.

In [8], a powerful tool is introduced to simplify and analyze complicated Markov chains with a large number of states. We will use the same methods to reduce the dimensions of the Markov chain defined in Lemma 2 as presented in the following corollaries.

Corollary 1 *The Markov chain $(I_S(t), I_R(t))$ can be collapsed into a new Markov chain $I_R(t)$ which represents the set of states of the form $(X, I_R(t))$, where $X = 0, 1, \ldots$.*

Corollary 2 *The Markov chain $(I_S(t), I_R(t))$ can be collapsed into a new Markov chain $I_S(t)$ which represents the set of states of the form $(I_S(t), Y)$, where $Y = 0, 1, \ldots, m$.*

Lemma 4 simplifies the condition of decodability introduced in Lemma 3 to include only the collapsed Markov chain $I_S(t)$ instead of the Markov chain $(I_S(t), I_R(t))$.

Lemma 4 *All the states of the Markov chain $(I_S(t), I_R(t))$ are recurrent if and only if all the states of the collapsed Markov chain $I_S(t)$ are recurrent.*

The proof can be found in Appendix D.

The following assumption is used to approximate the limit on the arrival rate λ. However, the assumption is not needed to prove the existence of such a limit.

Assumption 1 *Let $\Pr\{(I_S, I_R)\}$, $\Pr\{I_S\}$, $\Pr\{I_R\}$ be the steady-state probability distributions of the Markov chains $(I_S(t), I_R(t))$, $I_S(t)$, and $I_R(t)$, respectively. Then, the steady-state probability distributions of source and relay are independent of each other,* i.e., $\Pr\{(I_S, I_R)|I_S\} = \Pr\{I_R\}$, *and* $\Pr\{(I_S, I_R)|I_R\} = \Pr\{I_S\}$.

Finally, the following results summarizes the decodability condition in terms of the source arrival rate λ.

Lemma 5 *In the collapsed Markov chain $I_R(t)$, the steady state probability $\pi_R(m) = \lim_{t\to\infty} \Pr\{I_R(t) = m\}$ is a non-decreasing continuous function of λ, achieving its maximum, $\pi_R^{max}(m)$, when all the states in the collapsed Markov chain $I_S(t)$ are transient or null-recurrent.*

The proof can be found in Appendix E.

Theorem 1 *A stream of packets with source arrival rate λ is decodable if and only if $\lambda < C^*$, where C^* is the maximum throughput,* i.e., $C^* = \overline{\varepsilon}_1 \overline{\pi}_R^{max}(m)$.

The proof can be found in Appendix F.

IV. Decoding Delay

In section III, the existence of an upper limit to the decodable arrival rate λ is proved and derived. However, as mentioned before, decodability with no packet loss or corruption of the buffer contents, comes at the cost of decoding delay. In this section, we address the problem of finding analytical expressions for the average length of decoded blocks and its variations with arrival rate λ. The average length of a decoded block is a measure of decoding delay at the network since a larger decoded block implies a larger average packet delay.

First, we start with the familiar two-hop example and propose an upper bound on the average length of decoded blocks. Then, we will generalize the bound for a multi-hop line network.

A. Average Length of Decoded Blocks: Two-hop Line Network

Given a stream of packets is decodable, the Markov chain $I_S(t)$ is ergodic and therefore, has a steady-state probability distribution, denoted by $\pi_S(.)$, where $\pi_S(i) = \lim_{t\to\infty} \Pr\{I_S(t) = i\}$ for $i = 0, 1, 2, \ldots$. Further, the steady-state probability distribution for the Markov chain $I_R(t)$ is denoted by $\pi_R(.)$, where $\pi_R(i) = \lim_{t\to\infty} \Pr\{I_R(t) = i\}$ for $i = 0, 1, \ldots, m$. Finally, Theorem 2 provides an upper bound on the average length of a decoded block in a two-hop line network setting.

Lemma 6 *Let T_0^+ be the time to return to zero for the Markov chain $I_S(t)$,* i.e. $T_0^+ = \min\{t > t_0 : I_S(t) = I_S(t_0) = 0\}$. *Then, the expected time to return to zero at steady-state is $E\left[T_0^+\right] = {\pi_S}^{-1}(0)$.*

For the proof, See the proof of Lemma 5 in Chapter 2 of [9].

Lemma 7 *Let $P_R^{dec}(k)$ be the probability that right after $I_S(t)$ returns to zero at time t_0,* i.e. *$I_S(t_0) = 0$, a block of packets including only the original packets arrived at the source up to time t_0 is decoded, given $I_R(t_0) = k$. Then, we have the following for $k = 1, 2, \ldots, m$:*

$$P_R^{dec}(k) > \{\varepsilon_1 \overline{\varepsilon}_2\}^{k-1}\, \overline{\varepsilon}_2\, e^{k\varepsilon_1\varepsilon_2}.$$

The proof can be found in Appendix G.

Theorem 2 *Let $\pi_R^{rcv}(k)$ be the conditional steady-state probability that $I_R(t) = k$ right after the relay receives a packet given that the relay is not full before the packet arrives,* i.e. *$I_R < m$. Let l_{dec} be the random variable representing the length of a decoded block. Then, the following provides an upper bound for the average length of a decoded block:*

$$E\left[l_{dec}\right] < \lambda E\left[T_0^+\right] \left\{\sum_{k=1}^{m} \pi_R^{rcv}(k) P_R^{dec}(k)\right\}^{-1}. \quad (4)$$

The proof can be found in Appendix H.

B. Average Length of Decoded Blocks: Multi-hop Line Network

Here, we extend the results of SectionIV-A to a line network with h hops. The steady-state probability distribution for the Markov chain $I_j(t)$ corresponding to the relay v_j is denoted by $\pi_j(\cdot)$ for $j = 1, 2, \ldots, h$, where $\pi_j(i) = \lim_{t\to\infty} \Pr\{I_j(t) = i\}$.

Lemma 8 *Let $P_1^{dec}(k_1)$ be the probability that right after $I_S(t)$ returns to zero at time t_0,* i.e. *$I_S(t_0) = 0$, all the information required to decode the original packets arrived at the source up to time t_0 is passed to the relay node v_1, given $I_1(t_0) = k_1$. Similarly, Let $P_2^{dec}(k_2)$ be the probability that right after $I_1(t)$ becomes zero at a time t_1,* i.e. *$I_1(t_1) = 0$, all the required information to decode the original packets arrived at the source up to time t_0 is passed to the relay node v_2, given $I_2(t_1) = k_2$. Further, Let $P_3^{dec}(k_3), \cdots, P_{h-1}^{(k_{h-1})}$ be defined in a similar fashion, where h is the number of hops. Then, we have the following for $j = 1, 2, \ldots, h-1$ and $k_j = 1, 2, \ldots, m_i$:*

$$P_j^{dec}(k_j) > \{\varepsilon_j r_{j+1}\}^{k_j - 1}\, r_{j+1}\, e^{k_j \varepsilon_j \overline{r}_{j+1}},$$

where, $r_j = \overline{\varepsilon}_j \overline{\pi}_j(m_j)$.

The proof is omitted due to lack of space.

Theorem 3 *Let $\pi_j^{rcv}(k)$ be the conditional steady-state probability that $I_j(t) = k$ right after node v_j receives a packet given that the relay is not full before the packet arrives,* i.e. *$I_j < m$. Then, the following provides an upper bound for the average length of a decoded block in a line network of h hops:*

$$E\left[l_{dec}\right] < \lambda E\left[T_0^+\right] \prod_{j=1}^{h-1} \left\{\sum_{k=1}^{m_j} \pi_j^{rcv}(k) P_j^{dec}(k)\right\}^{-1}. \quad (5)$$

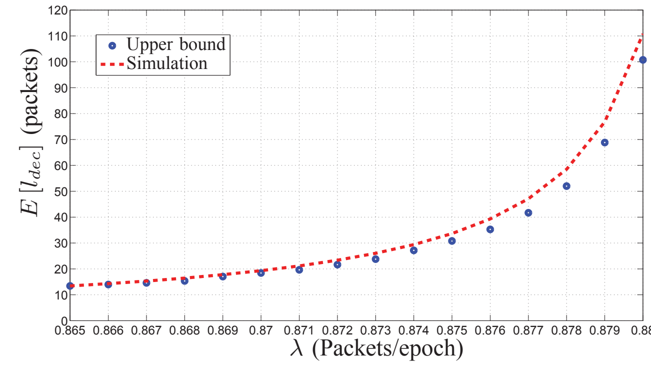

Fig. 1. Variations of the average length of a decoded block in a two-hop line network

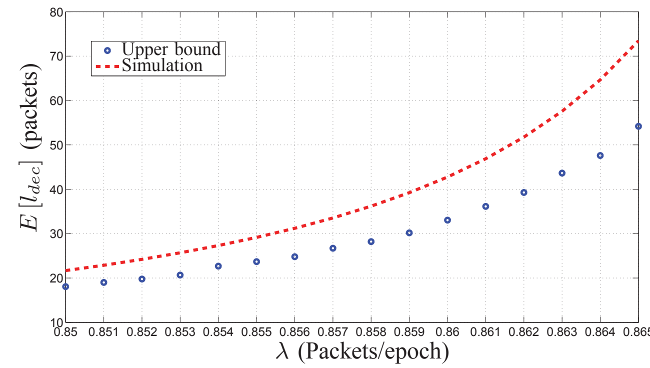

Fig. 2. Variations of the average length of a decoded block in a three-hop line network

The proof is omitted due to lack of space.

C. Simulation Results

In this section, the proposed upper bounds are validated by comparing it with simulations. In our simulation setup, the buffer size of all the relay nodes are assumed to be equal, $m = 5$ packets. Further, the probability of erasure on all the links are assumed to be the same, $\varepsilon = 0.1$. The mean arrival rate at the source, λ, is varied in a range that the stream of packets remain decodable. Then, the variations of the average length of a decoded block are presented in Fig. 1, Fig. 2, and Fig. 3 for a line network with 2, 3, and 4 hops, respectively. Clearly, from the simulation results, the upper bound is fairly tight for a two-hop line network, and as the number of hops increases, the upper bound becomes looser. The reason for such behavior is multiplication of the upper bound introduced for the two-hop case for a multi-hop scenario.

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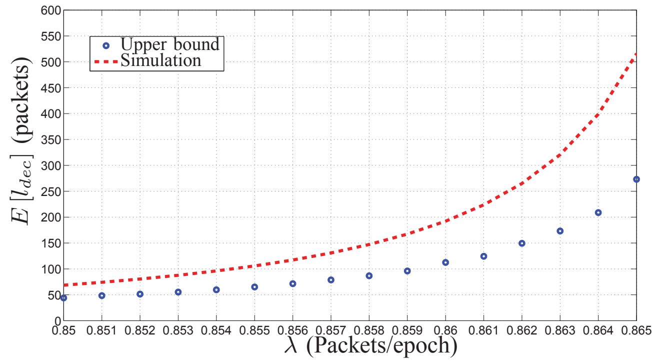

Fig. 3. Variations of the average length of a decoded block in a four-hop line network

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Appendix A
Proof of Lemma 1

Suppose that a block of length K is decoded at time t^*. Then, at time t^*, the number of equations at the destination must have become equal to the number of unknowns, *i.e.*, $I_D(t^*) = P_D(t^*)$. Further, by definition, t_0 is the last time that the event $P_D(t) = I_D(t)$ has occurred before t^*, hence, $P_D(t) > I_D(t)$ for $t_0 < t < t^*$. Therefore, $I_D(t^*) - I_D(t_0)$ is the number of equations in the latest solvable set of linear equations, leading to find $I_D(t^*) - I_D(t_0)$ unknowns. The length of the decoded block being K, results in $I_D(t^*) - I_D(t_0) = K$. The proof of the reverse statement is straightforward and follows the same steps as mentioned.

Appendix B
Proof of Lemma 2

Given the channel realizations at time t, *i.e.*, whether if a packet is lost or not at time t, and knowing the way the innovativeness of each node changes with arrival and/or departure of packets, it is clear that $(I_S(t), I_R(t))$ only depends on $(I_S(t-1), I_R(t-1))$.

Appendix C
Proof of Lemma 3

Suppose that a stream of packets with source arrival rate λ is decodable. Assume that all of the states of the form $(0, Y)$ are transient, where $Y = 0, 1, \ldots, m$. In this case, after a certain amount of time and also after the last block of packet is decoded, the Markov chain $(I_S(t), I_R(t))$ will never visit any of the states of the form $(0, Y)$. Hence, at no point in time the number of equations generated and transmitted at the source will be as many as the number of unknowns used and hence, no block of packets will be ever decoded from that certain time forward. Next, suppose that in the Markov chain $(I_S(t), I_R(t))$, any state of the form $(0, Y)$ is recurrent, where $Y = 0, 1, \ldots, m$. Then, after visiting an arbitrary state $(0, i)$, the block of packets will be decoded with the successful transmission of i packets to the destination without receiving more packets from the source. Since i is finite ($i \leq m$), this event happens with a positive probability. Since a return to such states of the form $(0, Y)$ is recurrent, in a finite time the block of packets will be decoded.

Appendix D
Proof of Lemma 4

Suppose all the states of the collapsed Markov chain $I_S(t)$ are recurrent. Assuming an arbitrary state $(I_S, I_R) = (i, j)$ in the Markov chain $(I_S(t), I_R(t))$ is transient means that there is a non-zero probability that $(I_S(t), I_R(t))$ will never return to the state (i, j). Using Theorem 2 of [8], the sum of the steady-state probabilities of the group of states of the form (i, Y) in the Markov chain $(I_S(t), I_R(t))$ is equal to the steady-state probability of the state i in the collapsed Markov chain $I_S(t)$, where $Y = 0, 1, \ldots, m$. The steady-state probability of the state i in the collapsed Markov chain $I_S(t)$ is non-zero since all the states of the collapsed Markov chain $I_S(t)$ are recurrent. Hence, there is at least one state of the form (i, Y), in the Markov chain $(I_S(t), I_R(t))$ has a non-zero steady-state probability. Suppose that the state (i, k) in the Markov chain $(I_S(t), I_R(t))$ has a non-zero steady state probability. Because of the structure of The the Markov chain $(I_S(t), I_R(t))$, the state (i, j) can be reached from the state (i, k) with a positive probability, *e.g.* by sending or receiving packets at the relay. Therefore, the state (i, j) is not transient and the result follows from this contradiction. The reverse direction is easy and can be done in a similar way.

Appendix E
Proof of Lemma 5

Assumption 1 implies that the steady state probability $\pi_R(m)$ equals the blocking probability that the source S perceives from the relay node R which is $\varepsilon_2 \Pr\{X_R = m\}$, where $\Pr\{X_R = m\}$ is the steady-state probability of the Markov chain corresponding to the relay [7]. Since ε_2 is assumed to be a constant, to prove the lemma, we need to prove the results for $\Pr\{X_R = m\}$ instead of $\pi_R(m)$. In the Markov chain corresponding to the relay, $\alpha = r_{in}\varepsilon_2$, $\beta = \overline{r_{in}\varepsilon_2}$, and $\alpha_0 = r_{in}$, where r_{in} is the arrival rate of packets from the source. Clearly, r_{in} increases with λ because larger λ increases the probability of the source to be non-empty and hence increases the arrival rate of innovative packets to the relay from the source. Therefore, α and α_0 increase with λ and β decreases with λ. Hence, $\Pr\{X_R = m\}$ is a non-decreasing function of λ and it can be seen that it is a continuous function as well [7]. Further, by increasing λ the probability of the source being in empty state decreases. However, at some point increasing λ leads to a situation in which the empty state of the source becomes transient or null-recurrent. In this case, the parameters α, α_0 and β will not change anymore and $\pi_R(m)$ achieves its maximum $\pi_R^{max}(m)$.

APPENDIX F
PROOF OF THEOREM 1

Using Lemma 3 and Lemma 4, it is clear that to prove the theorem we need to prove that the state 0 in the collapsed Markov chain $I_S(t)$ is recurrent if and only if $\lambda < C^*$, where $C^* = \overline{\varepsilon}_1 \overline{\pi}_R^{max}(m)$.

Suppose that former holds, *i.e.*, the state 0 in the Markov chain $I_S(t)$ is recurrent. Then, assume that $\lambda \geq C^*$. Further, let $r_{out}(\lambda)$ be the maximum possible departure rate at the source which equals $\overline{\varepsilon}_1 \overline{\pi}_R(m)$. From Lemma 5, we know that $\pi_R(m)$ is a non-decreasing continuous function of λ, achieving its maximum, $\pi_R^{max}(m)$, when all the states in the collapsed Markov chain $I_S(t)$ are transient or null-recurrent. Hence, $r_{out}(\lambda)$ is a non-increasing continuous function of λ, achieving its minimum, $r_{out}^{min} = \overline{\varepsilon}_1 \overline{\pi}_R^{max}(m) = C^*$, when all the states in the collapsed Markov chain $I_S(t)$ are transient or null-recurrent. Since the state 0 in the Markov chain $I_S(t)$ is recurrent, it is clear that the arrival rate at the source is smaller than the maximum possible departure rate, *i.e.*, $\lambda < r_{out}(\lambda)$. It is also known that $r_{out}(\lambda) \geq C^*$ since $C^* = r_{out}^{min}$. Let λ^* be the smallest arrival rate at the source for which the state 0 of the Markov chain $I_S(t)$ is transient or null-recurrent meaning for any arrival rate smaller than λ^* the state 0 is recurrent, *i.e.*, $\lambda < r_{out}(\lambda)$ for any $\lambda < \lambda^*$. Then, because $r_{out}(\lambda)$ is a continuous function of λ, we have $\lambda^* = r_{out}(\lambda^*)$. Further, $r_{out}(\lambda^*) = r_{out}^{min} = C^*$ because $r_{out}(\cdot)$ achieves its minimum when all the states in the Markov chain $I_S(t)$ are transient or null-recurrent. Note that, if state 0 is transient, then every other state in $I_S(t)$ is also transient. Hence, we have $\lambda^* = C^*$ and consequently, $\lambda < C^*$ which is a contradiction to the assumption $\lambda \geq C^*$. Therefore, the assumption $\lambda \geq C^*$ must be false which proves the results.

The proof of the reverse is straightforward. Assuming $\lambda < C^*$ guarantees that the state 0 in the Markov chain $I_S(t)$ is recurrent since $C^* = \overline{\varepsilon}_1 \overline{\pi}_R^{max}(m)$ is the minimum of the maximum possible departure rates at the source and hence guarantees that the arrival rate λ is smaller than any maximum departure rates at the source.

APPENDIX G
PROOF OF LEMMA 7

First, we need to find the condition for decoding the packets arrived at the source up to time t_0. Right after $I_S(t)$ becomes zero, all the needed useful equations for the destination to decode the packets arrived at the source up to time t_0 are now stored at the relay node. Further, $I_R(t_0) = k$ implies that there are only k of such equations available at the relay node. Therefore, to be able to decode, the relay node should not receive any innovative packet from the source while the destination is receiving k packets from the relay. Let δ be the probability of the event that in a single time epoch source transmits a packet and the packet is either lost or not innovative for the relay. Since the source is empty at t_0, there is a higher chance that the source remains empty at the next few epochs, leading to $\delta = 1$. However, after a few epochs, a packet arrives at the source and we have $\delta = \varepsilon_1$. Hence, assuming $\delta \geq \varepsilon_1$ is a reasonable approximation for the purpose of steady-state analysis. Consider the scenario in which the task of decoding will be completed in exactly $k+i$ epochs[4], where $i = 0, 1, 2, \ldots$. We proceed to compute the probability of this scenario. In i of the epochs from the first $i+k-1$ epochs, at the relay, neither an innovative packet should be received nor a packet should be successfully transmitted, which happens with probability $\delta\varepsilon_2$ in a single epoch. Further, in $k-1$ of the epochs from the first $i+k-1$ epochs, At the relay, a packet has to be successfully transmitted to the destination while no packet arrives from the source, which happens with probability $\delta\overline{\varepsilon}_2$ in a single epoch. Finally, in the last epoch, a packet has to be received by the destination, which happens with probability $\overline{\varepsilon}_2$. Therefore, we have the following:

$$
\begin{aligned}
P_R^{dec}(k) &= \sum_{i=0}^{\infty} \binom{k+i-1}{i} \{\delta\varepsilon_2\}^i \{\delta\overline{\varepsilon}_2\}^{k-1} \overline{\varepsilon}_2 && (6)\\
&\geq \sum_{i=0}^{\infty} \binom{k+i-1}{i} \{\varepsilon_1\varepsilon_2\}^i \{\varepsilon_1\overline{\varepsilon}_2\}^{k-1} \overline{\varepsilon}_2 && (7)\\
&= \{\varepsilon_1\overline{\varepsilon}_2\}^{k-1} \overline{\varepsilon}_2 \sum_{i=0}^{\infty} \frac{(k+i-1)\cdots(k)}{i!} \{\varepsilon_1\varepsilon_2\}^i && (8)\\
&> \{\varepsilon_1\overline{\varepsilon}_2\}^{k-1} \overline{\varepsilon}_2 \sum_{i=0}^{\infty} \frac{k^i}{i!} \{\varepsilon_1\varepsilon_2\}^i && (9)\\
&= \{\varepsilon_1\overline{\varepsilon}_2\}^{k-1} \overline{\varepsilon}_2 e^{k\varepsilon_1\varepsilon_2} && (10)
\end{aligned}
$$

Note that, (7) is the result of assuming $\delta \geq \varepsilon_1$.

APPENDIX H
PROOF OF THEOREM 2

Before any block of packets is decoded at the destination, the following events must occur: $I_S(t')$ returns to the state 0, and $I_R(t') = k$ with probability $\pi_R^{rcv}(k)$, where $k = 1, 2, \ldots, m$. For each k, all the packets arrived at the source up to time t' will be decoded with probability $P_R^{dec}(k)$. Therefore, every time $I_S(t)$ returns to zero at epoch t', all the packets arrived at the source up to time t' will be decoded with the average probability $\sum_{k=1}^{m} \pi_R^{rcv}(k) P_R^{dec}(k)$. Further, since the expected waiting time for $I_S(t)$ to return to zero is $E\left[T_0^+\right]$, the average time it takes for a block of packets to be decoded at the destination is $E\left[T_0^+\right]\left\{\sum_{k=1}^{m} \pi_R^{rcv}(k) P_R^{dec}(k)\right\}^{-1}$. Finally, the rate at which the destination receives innovative packets is λ given that the Markov chain $I_S(t)$ is ergodic, which is the case since we assume the stream is decodable. Hence, $\lambda E\left[T_0^+\right]\left\{\sum_{k=1}^{m} \pi_R^{rcv}(k) P_R^{dec}(k)\right\}^{-1}$ will be the average length of a decoded block, and the results follows.

[4]The minimum number of epochs to complete the decoding is k.