ECE4270
Fundamentals of DSP

Lecture 2
Discrete-Time Signals and Systems
&
Difference Equations

School of ECE
Center for Signal and Information Processing
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Overview of Lecture 2

- Announcement
- Discrete-time systems (from Lecture #1)
  - Properties
  - Testing for properties
- Linear time-invariant systems (LTI) (from Lecture #1)
  - Convolution sum
- Example of evaluation of discrete convolution
- Stability of LTI systems
- Causality of LTI systems
- Cascade and parallel connections of LTI systems
- Difference equations
- Initial rest conditions and LTI

Announcement

- Homework will be assigned weekly on Thursdays and will be due the following Thursdays.

Link to the class web page:

http://fekri.ece.gatech.edu/course_ece4270.html

More Discrete-Time Systems

$x[n]$ → **D-T System** → $y[n] = \mathrm{T}\{x[n]\}$

- *L*-point moving average system:

$$\begin{aligned} y[n] &= \frac{1}{L}\sum_{k=0}^{L-1} x[n-k] \\ &= \frac{1}{L}(x[n] + x[n-1] + \cdots + x[n-L+1]) \end{aligned}$$

- Accumulator system:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Properties of D-T Systems

- A system is linear if and only if

$$T\{ax_1[n]+bx_2[n]\}=aT\{x_1[n]\}+bT\{x_2[n]\}$$

- A system is time-invariant if and only if

$$x_1[n]=x[n-n_d] \quad \Rightarrow \quad y_1[n]=y[n-n_d]$$

- A system is causal if and only if

$$y[n] \text{ depends only on } x[k] \text{ for } k \le n$$

- A system is BIBO stable if every bounded input produces a bounded output; i.e.,

$$\text{when } |x[n]| < B_x < \infty \text{ for all } n,$$
$$\text{then } |y[n]| < B_y < \infty \text{ for all } n$$

Moving Averager: $y[n]=(1/L)\sum_{k=0}^{L-1} x[n-k]$

- Linearity: yes

$$\frac{1}{L}\sum_{k=0}^{L-1}(ax_1[n-k]+bx_2[n-k])=a\left(\frac{1}{L}\sum_{k=0}^{L-1}x_1[n-k]\right)+b\left(\frac{1}{L}\sum_{k=0}^{L-1}x_2[n-k]\right)$$

- Time-invariance: yes

$$\frac{1}{L}\sum_{k=0}^{L-1}x[n-k-n_d]=\frac{1}{L}\sum_{k=0}^{L-1}x[(n-n_d)-k]=y[n-n_d]$$

- Causality: yes

$$y[n]=\frac{1}{L}(x[n]+x[n-1]+\cdots+x[n-L+1])$$

- Stability: yes

$$|y[n]|=\left|\frac{1}{L}\sum_{k=0}^{L-1}x[n-k]\right|\le\frac{1}{L}\sum_{k=0}^{L-1}|x[n-k]|\le B_x$$

Down Sampler: $y[n]=x[Mn]$

- Linearity yes

$$ax_1[Mn]+bx_2[Mn]=ay_1[n]+by_2[n]$$

- Time-invariance no

$$y_1[n]=x_1[Mn]=x[Mn-n_d]\ne y[n-n_d]=x[M(n-n_d)]$$

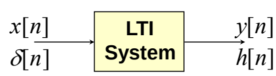
- Causality no

$$y[-1]=x[-M], \text{ but } y[+1]=y[M]$$

- Stability yes

$$|y[n]|=|x[Mn]|\le B_x$$

LTI Discrete-Time Systems



- Linearity (superposition):

$$T\{ax_1[n]+bx_2[n]\}=aT\{x_1[n]\}+bT\{x_2[n]\}$$

- Time-Invariance (shift-invariance):

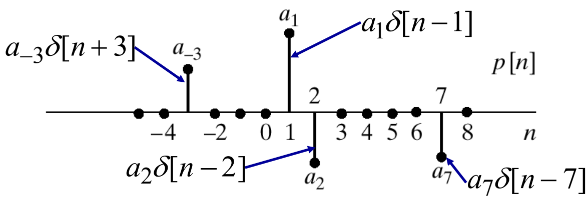
$$x_1[n]=x[n-n_d] \quad \Rightarrow \quad y_1[n]=y[n-n_d]$$

- LTI implies discrete convolution:

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=x[n]*h[n]=h[n]*x[n]$$

Impulse Representation of Sequences

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



$$p[n] = a_{-3}\delta[n+3] + a_1\delta[n-1]$$
$$+ a_2\delta[n-2] + a_7\delta[n-7]$$

LTI Discrete-Time Systems

$x[n]$ → **LTI System** → $y[n]$

"impulse" $\delta[n]$ → $h[n]$ impulse response

$x[k]\delta[n-k]$ → $x[k]h[n-k]$

$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n]$ → $\sum_{k=-\infty}^{\infty} x[k]h[n-k] = y[n]$

convolution sum

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n]*h[n] = h[n]*x[n]$$

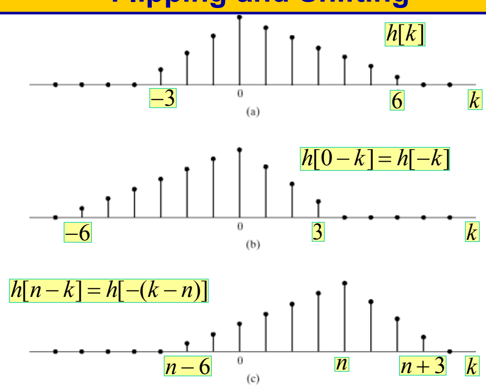
Discrete Convolution

- Two ways to look at it:
  - As the representation of the output as a sum of delayed and scaled impulse responses.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[0]h[n] + x[1]h[n-1] + \ldots$$
$$+ x[-1]h[n+1] + \ldots$$

  - As a computational formula for computing $y[n]$ ("y at time n") from the entire sequences x and h.
    - Form $x[k]h[n-k]$ for $-\infty < k < \infty$ for fixed n.
    - Sum over all k to produce $y[n]$.
    - Repeat for all n.

"Flipping and Shifting"



Discrete Convolution - I

- Definition

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]$$

- Example

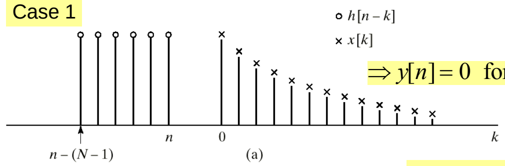
$$h[n]=\begin{cases}1, & 0\le n\le 5\\ 0, & \text{otherwise}\end{cases}$$

$$y[n]=\sum_{k=n-5}^{n}x[k]=\sum_{k=0}^{5}x[n-k]$$

$$y[n]=x[n]+x[n-1]+\ldots+x[n-5]$$

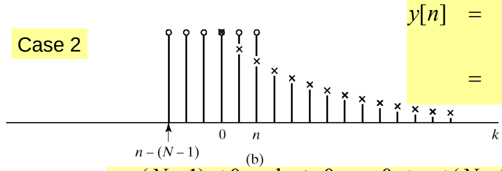
Discrete Convolution - II

Case 1



$$\Rightarrow y[n]=0 \ \text{ for } n<0$$

Case 2

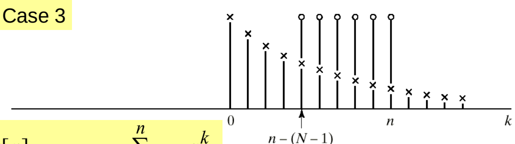


$$y[n]=\sum_{k=0}^{n}a^k=\frac{1-a^{n+1}}{1-a}$$

$$n-(N-1)\le 0 \text{ and } n\ge 0 \quad\text{or}\quad 0\le n\le (N-1)$$

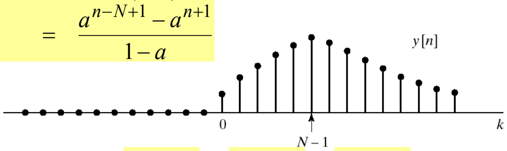
Discrete Convolution - III

Case 3



$$y[n]=\sum_{k=n-(N-1)}^{n}a^k=\frac{a^{n-N+1}-a^{n+1}}{1-a}$$

$$n-(N-1)>0 \quad\text{or}\quad n>(N-1)$$



Case 1 Case 2 Case 3

Stability of LTI Systems

- Stability: Every bounded input produces a bounded output.

$$|y[n]|=\left|\sum_{k=-\infty}^{\infty}h[k]x[n-k]\right|\le\sum_{k=-\infty}^{\infty}|h[k]||x[n-k]|$$

$$|y[n]|\le\sum_{k=-\infty}^{\infty}|h[k]|B_x$$

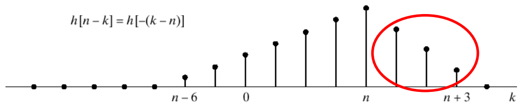
$$\text{Therefore, } |y[n]|<\infty \text{ if } \quad \sum_{k=-\infty}^{\infty}|h[k]|<\infty$$

- This condition can also be shown to be necessary as well as sufficient for BIBO stability.

Causality of LTI Systems

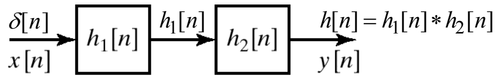
- A system is causal if $y[n]$ depends only on $x[k]$ for k less than or equal to n.

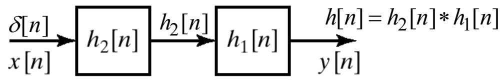
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \Rightarrow h[n-k] = 0 \quad \text{for} \quad k > n$$



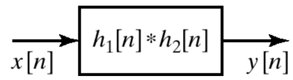
Causality requires $h[n] = 0 \quad \text{for} \quad n < 0$

Cascade Connection of LTI Systems



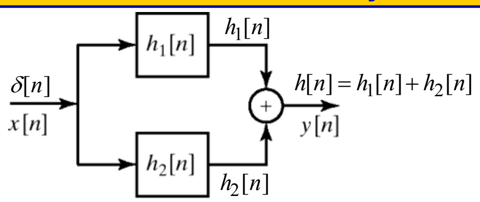


Since $h_1[n] * h_2[n] = h_2[n] * h_1[n]$, it follows that



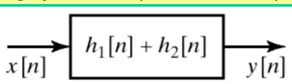
is equivalent to the cascade in either order.

Parallel Combination of LTI Systems



(a)

The following system is equivalent to the parallel combination.



(b)

Examples

- Delay: $y[n] = x[n - n_d]$

$$h[n] = \delta[n - n_d] \Rightarrow x[n] * \delta[n - n_d] = x[n - n_d]$$

- Accumulator: $y[n] = \sum_{k=-\infty}^{n} x[k]$

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases} = u[n]$$

- First difference: $y[n] = x[n] - x[n-1]$

$$h[n] = \delta[n] - \delta[n-1]$$

Difference Equations

- For all computationally realizable LTI systems, the input and output satisfy a difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- This leads to the recurrence formula

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right) y[n-k] + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right) x[n-k]$$

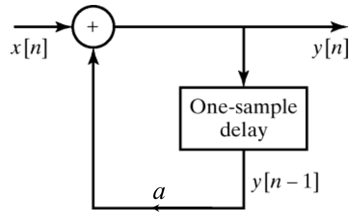
which can be used to compute the “present” output from the present and *M* past values of the input and *N* past values of the output

First-Order Example

- Consider the difference equation

$$y[n] = ay[n-1] + x[n]$$

- We can represent this system by the following block diagram:



Recursive Computation of Output

Let $x[n] = K\delta[n]$ and $y[-1] = c$.

n	$x[n]$	$y[n] = ay[n-1] + x[n]$
…	…	…
-1	0	c
0	K	$ac + K$
1	0	$a(ac+K) = a^2c + Ka$
2	0	$a(a^2c + Ka) = a^3c + Ka^2$
3	0	$a(a^3c + Ka^2) = a^4c + Ka^3$
…	…	…

General Solution

- By induction, we see that if

$$y[n] = ay[n-1] + x[n]$$

with $x[n] = K\delta[n]$ and $y[-1] = c$.

then the solution is

$$y[n] = ca^{n+1} + Ka^n \quad \text{for} \quad n \geq -1$$

- If $x[n] = K\delta[n-n_d]$, then the output will be

$$y[n] = ca^{n+1} + Ka^{n-n_d} \quad \text{for} \quad n \geq -1$$ ← Implies Not TI

- If $x[n] = \delta[n]$, then the output will be

$$y[n] = ca^{n+1} + a^n \quad \text{for} \quad n \geq -1$$ ← Implies Not linear

LTI Recursive Implementation

- We say that an input is *suddenly applied at time n_d* if x[n]=0 for all $n<n_d$.
- If the input is suddenly applied and we assume that y[n]=0 for all $n<n_d$, then the iterative computation will be both linear and time-invariant. This assumption provides the required set of auxiliary conditions $\{y[n_d-1], y[n_d-2], , \ldots, y[n_d-N]\}$ that is required to get the recursion going. Zero auxiliary conditions are called the *initial rest conditions*.
- For the first-order case of our example, the impulse response of the LTI system is $h[n]=a^n u[n]$

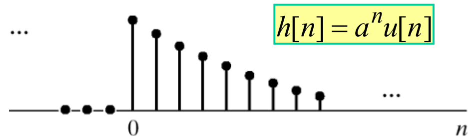
Exponential Impulse Response

- With initial rest conditions, the difference equation

$$y[n]=ay[n-1]+x[n]$$

has impulse response

$$h[n]=a^n u[n]$$



LTI Recursion of First-Order DE

- If we assume initial rest conditions, then the difference equation $y[n]=ay[n-1]+x[n]$

  has impulse response $h[n]=a^n u[n]$

- In other words, y[n] is also given by the convolution

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]a^{n-k}u[n-k]=\sum_{k=-\infty}^{n} x[k]a^{n-k}$$