ECE4270 Fundamentals of DSP

Lecture 3 Frequency Response of LTI Systems

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Overview of Lecture 3

- Announcement
- Difference equations (from Last Lecture Slides)
- Initial rest conditions and LTI (from Last Lecture)
- IIR systems
- FIR systems
- Matlab and LTI systems
- Complex exponential inputs to LTI systems
- The frequency response
- Examples
  - Delay, First difference, Moving average
- Plotting the frequency response

Announcement

- TA information:
  - Yashas Saidutta
  - Email: <ysaidutta3@gatech.edu>
  - office hours:
    - Thursday 9-10
    - MW 4:30 PM-5:30 PM
    - Location: VL C449. Fourth floor of Van Leer Building, Central corridor.

Difference Equations

- For all computationally realizable LTI systems, the input and output satisfy a difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- This leads to the recurrence formula

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)y[n-k] + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-k]$$

which can be used to compute the "present" output from the present and *M* past values of the input and *N* past values of the output

Recursive Computation of Output

Let $x[n] = K\delta[n]$ and $y[-1] = c$.

n	$x[n]$	$y[n] = ay[n-1] + x[n]$
$\cdots$	$\cdots$	$\cdots$
-1	0	c
0	K	$ac + K$
1	0	$a(ac+K) = a^2c + Ka$
2	0	$a(a^2c + Ka) = a^3c + Ka^2$
3	0	$a(a^3c + Ka^2) = a^4c + Ka^3$
$\cdots$	$\cdots$	$\cdots$

General Solution

- By induction, we see that if

$$y[n] = ay[n-1] + x[n]$$

with $x[n] = K\delta[n]$ and $y[-1] = c$.

then the solution is

$$y[n] = ca^{n+1} + Ka^n \quad \text{for} \quad n \geq -1$$

- If $x[n] = K\delta[n - n_d]$, then the output will be

$$y[n] = ca^{n+1} + Ka^{n-n_d} \quad \text{for} \quad n \geq -1$$

Implies Not TI

- If $x[n] = \delta[n]$, then the output will be

$$y[n] = ca^{n+1} + a^n \quad \text{for} \quad n \geq -1$$

Implies Not linear

LTI Recursive Implementation

- We say that an input is *suddenly applied at time n_d* if x[n]=0 for all $n<n_d$.
- If the input is suddenly applied and we assume that y[n]=0 for all $n<n_d$, then the iterative computation will be both linear and time-invariant. This assumption provides the required set of auxiliary conditions {$y[n_d-1]$, $y[n_d-2]$, , …, $y[n_d-N]$} that is required to get the recursion going. Zero auxiliary conditions are called the *initial rest conditions*.
- For the first-order case of our example, the impulse response of the LTI system is $\boxed{h[n] = a^n u[n]}$

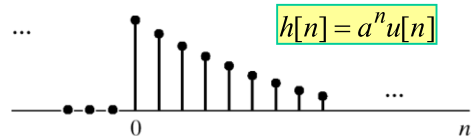
Exponential Impulse Response

- With initial rest conditions, the difference equation

$$y[n] = ay[n-1] + x[n]$$

has impulse response

$$\boxed{h[n] = a^n u[n]}$$



LTI Recursion of First-Order DE

- If we assume initial rest conditions, then the difference equation $y[n] = ay[n-1] + x[n]$

  has impulse response $\boxed{h[n] = a^n u[n]}$

- In other words, y[n] is also given by the convolution

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k] a^{n-k} u[n-k] = \sum_{k=-\infty}^{n} x[k] a^{n-k}}$$

IIR Systems

- Under conditions of initial rest, a system whose input and output satisfy a difference equation of the form

$$y[n] = \underbrace{\sum_{k=1}^{N} a_k y[n-k]}_{\text{feedback}} + \underbrace{\sum_{k=0}^{M} b_k x[n-k]}_{\text{feedforward}}$$

Note the redefinition of the coefficients.

is LTI and its impulse response is of the form

$$h[n] = \sum_{k=1}^{N} A_k \alpha_k^n u[n] = \begin{cases} A_k \alpha_k^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$\Rightarrow$ Infinite duration Impulse Response (IIR)

FIR Systems

- If there are no feedback terms, then the difference equation becomes

$$\boxed{y[n] = \sum_{k=0}^{M} b_k x[n-k]}$$

- This system is LTI and its impulse response is

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$ Finite duration Impulse Response (FIR)

Moving Average System

$$\boxed{y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]}$$

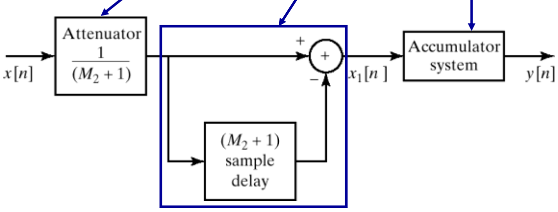
$$h[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} \delta[n-k] = \frac{1}{M_2 + 1} \begin{cases} 1 & 0 \leq n \leq M_2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{M_2 + 1} (u[n] - u[n - M_2 - 1])$$

$$= \frac{1}{M_2 + 1} (\delta[n] - \delta[n - M_2 - 1]) * u[n]$$

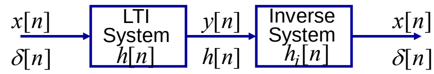
Moving Average System

$$h[n] = \underbrace{\left(\frac{1}{M_2+1}\right)}\bullet\underbrace{(\delta[n]-\delta[n-M_2-1])}*\underbrace{u[n]}$$



Inverse Systems

- An inverse system compensates (undoes) the effects of another system.



$$\boxed{\Rightarrow h[n]*h_i[n]=\delta[n]}$$

- The accumulator and first-difference systems are inverses of each other.

$$(\delta[n]-\delta[n-1])*u[n]=u[n]-u[n-1]=\delta[n]$$

- Understanding inverse systems is greatly facilitated by transform methods.

MATLAB and LTI Systems

```
»help conv
 CONV Convolution and polynomial multiplication.
   Y = CONV(X, H) convolves vectors X and H.  The resulting
   vector is length LENGTH(X)+LENGTH(H)-1.
   If X and H are vectors of polynomial coefficients, convolving
   them is equivalent to multiplying the two polynomials.
```

```
»help filter
 FILTER One-dimensional digital filter.
   Y = FILTER(B,A,X) filters the data in vector X with the
   filter described by vectors A and B to create the filtered
   data Y.  The filter is a "Direct Form II Transposed"
   implementation of the standard difference equation:
```


MATLAB and LTI Systems

- The moving average system

  >> h=ones(1,M+1)/(M+1);

  >> y=conv(x,h);

- The accumulator system:

  >> b=1; a=[1,-1];

  >> y=filter(b,a,x);

Complex Exponential Input Signals

$x[n] \rightarrow$ LTI System $\rightarrow y[n]$

$\delta[n] \qquad h[n]$

$e^{j\omega n} \qquad H(e^{j\omega})e^{j\omega n}$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}e^{j\omega n}$$

$$= H(e^{j\omega})e^{j\omega n}$$

Frequency response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

The Frequency Response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Periodicity of the frequency response

$$H(e^{j(\omega+2\pi)}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j(\omega+2\pi)k} = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\cancelto{1}{e^{-j2\pi k}}$$

- Existence of the frequency response

$$\left|H(e^{j\omega})\right| = \left|\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right| \le \sum_{k=-\infty}^{\infty} |h[k]|\cancelto{1}{\left|e^{-j\omega k}\right|}$$

$$\left|H(e^{j\omega})\right| \le \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Same as condition for stability!

Examples

- Delay: $y[n] = x[n-n_d]$

$$h[n] = \delta[n-n_d] \Rightarrow x[n] * \delta[n-n_d] = x[n-n_d]$$

- Moving Average: $y[n] = \dfrac{1}{M+1}\displaystyle\sum_{k=0}^{M} x[n-k]$

$$h[n] = \frac{1}{M+1}\sum_{k=0}^{M} \delta[n-k]$$

- First difference: $y[n] = x[n] - x[n-1]$

$$h[n] = \delta[n] - \delta[n-1]$$

Delay and First Difference

- Delay: $y[n] = x[n-n_d]$ $\quad h[n] = \delta[n-n_d]$

$$x[n] = e^{j\omega n} \mapsto y[n] = e^{j\omega(n-n_d)} = \underbrace{e^{-j\omega n_d}}_{H(e^{j\omega})} e^{j\omega n}$$

$$H(e^{j\omega}) = e^{-j\omega n_d}$$

- First difference:

$$y[n] = x[n] - x[n-1] \qquad h[n] = \delta[n] - \delta[n-1]$$

$$x[n] = e^{j\omega n} \mapsto y[n] = e^{j\omega n} - e^{j\omega(n-1)} = \underbrace{(1-e^{-j\omega})}_{H(e^{j\omega})} e^{j\omega n}$$

$$H(e^{j\omega}) = 1 - e^{-j\omega}$$

Moving Average

$$\boxed{y[n] = \frac{1}{M+1}\sum_{k=0}^{M} x[n-k]}$$

$$x[n] = e^{j\omega n} \mapsto y[n] = \frac{1}{M+1}\sum_{k=0}^{M} e^{j\omega(n-k)}$$

$$= \underbrace{\frac{1}{M+1}\sum_{k=0}^{M} e^{-j\omega k}}_{H(e^{j\omega})} e^{j\omega n}$$

$$\boxed{H(e^{j\omega}) = \frac{1}{M+1}\sum_{k=0}^{M} e^{-j\omega k}}$$

Moving Average Frequency Response

$$H(e^{j\omega}) = \frac{1}{M+1}\sum_{k=0}^{M} e^{-j\omega k} = \frac{1}{M+1}\frac{(1-e^{-j\omega(M+1)})}{(1-e^{-j\omega})}$$

$$H(e^{j\omega}) = \frac{1}{M+1}\frac{(e^{j\omega(M+1)/2} - e^{-j\omega(M+1)/2})e^{-j\omega(M+1)/2}}{(e^{j\omega/2} - e^{-j\omega/2})e^{-j\omega/2}}$$

$$\boxed{H(e^{j\omega}) = \frac{1}{M+1}\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}}$$

Plotting using MATLAB

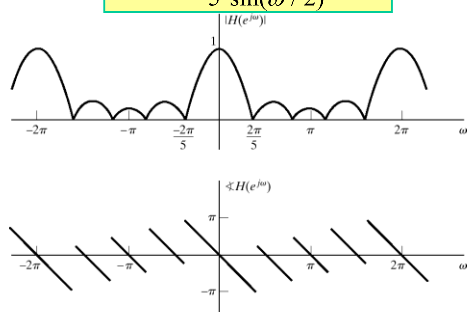
»help freqz

FREQZ Z-transform digital filter frequency response.
When N is an integer, [H,W] = FREQZ(B,A,N) returns the N-point frequency vector W in radians and the N-point complex frequency response vector H given numerator and denominator coefficients in vectors B and A. The frequency response is evaluated at N points equally spaced around the upper half of the unit circle. If N isn't specified, it defaults to 512.

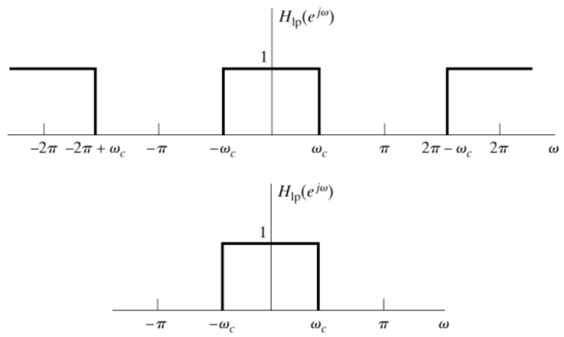
```
>> omega=(0:400)*pi/400; b=[1,1,1,1,1]/5;
>> H=freqz(b,1,omega);
>> subplot(211); plot(omega/pi,abs(H))
>> subplot(212); plot(omega/pi,angle(H))
```


Plotting the Frequency Response

$$\boxed{H(e^{j\omega}) = \frac{1}{5}\frac{\sin[\omega 5/2]}{\sin(\omega/2)}e^{-j\omega 2}} \qquad \boxed{M=4}$$



Ideal Lowpass Filter



Frequency Selective Filters