ECE4270
Fundamentals of DSP

Lecture 4

Discrete-Time Fourier Transform

School of ECE
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Overview of Lecture 4

- Examples of Freq. Response (Last Lecture)
- Plotting the frequency response (Last Lecture)
- Response of an LTI system to a cosine signal
- Definition of the discrete-time Fourier transform
- The frequency response as DTFT of the impulse response
- Ideal Lowpass filter
- Examples of DTFT
- Symmetry properties of the DTFT
- Examples of usage of the DTFT

Last Lecture: Complex Exponential Input Signals

$$\begin{array}{ccc} x[n] & & y[n] \\ \delta[n] & \longrightarrow \boxed{\text{LTI System}} \longrightarrow & h[n] \\ e^{j\omega n} & & H(e^{j\omega})e^{j\omega n} \end{array}$$

Frequency response

$$\boxed{H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}}$$

Examples

- Delay: $\boxed{y[n] = x[n-n_d]}$

  $h[n] = \delta[n-n_d] \Rightarrow x[n] * \delta[n-n_d] = x[n-n_d]$

- Moving Average: $\boxed{y[n] = \frac{1}{M+1}\sum_{k=0}^{M} x[n-k]}$

  $$\boxed{h[n] = \frac{1}{M+1}\sum_{k=0}^{M} \delta[n-k]}$$

- First difference: $\boxed{y[n] = x[n] - x[n-1]}$

  $$\boxed{h[n] = \delta[n] - \delta[n-1]}$$

Delay and First Difference

- Delay: $\boxed{y[n] = x[n-n_d]}$ $\boxed{h[n] = \delta[n-n_d]}$

$$x[n] = e^{j\omega n} \mapsto y[n] = e^{j\omega(n-n_d)} = \underbrace{e^{-j\omega n_d}}_{H(e^{j\omega})} e^{j\omega n}$$

$$\boxed{H(e^{j\omega}) = e^{-j\omega n_d}}$$

- First difference:

$\boxed{y[n] = x[n] - x[n-1]}$ $\boxed{h[n] = \delta[n] - \delta[n-1]}$

$$x[n] = e^{j\omega n} \mapsto y[n] = e^{j\omega n} - e^{j\omega(n-1)} = \underbrace{(1-e^{-j\omega})}_{H(e^{j\omega})} e^{j\omega n}$$

$$\boxed{H(e^{j\omega}) = 1 - e^{-j\omega}}$$

Moving Average

$$\boxed{y[n] = \frac{1}{M+1}\sum_{k=0}^{M} x[n-k]}$$

$$x[n] = e^{j\omega n} \mapsto y[n] = \frac{1}{M+1}\sum_{k=0}^{M} e^{j\omega(n-k)}$$

$$= \underbrace{\frac{1}{M+1}\sum_{k=0}^{M} e^{-j\omega k}}_{H(e^{j\omega})} e^{j\omega n}$$

$$\boxed{H(e^{j\omega}) = \frac{1}{M+1}\sum_{k=0}^{M} e^{-j\omega k}}$$

Moving Average Frequency Response

$$H(e^{j\omega}) = \frac{1}{M+1}\sum_{k=0}^{M} e^{-j\omega k} = \frac{1}{M+1}\frac{(1-e^{-j\omega(M+1)})}{(1-e^{-j\omega})}$$

$$H(e^{j\omega}) = \frac{1}{M+1}\frac{(e^{j\omega(M+1)/2} - e^{-j\omega(M+1)/2})e^{-j\omega(M+1)/2}}{(e^{j\omega/2} - e^{-j\omega/2})e^{-j\omega/2}}$$

$$\boxed{H(e^{j\omega}) = \frac{1}{M+1}\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}}$$

Plotting using MATLAB

»help freqz

FREQZ Z-transform digital filter frequency response.

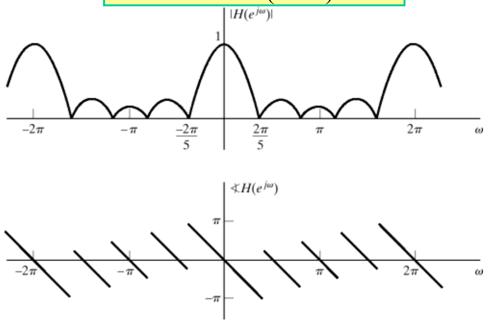
When N is an integer, [H,W] = FREQZ(B,A,N) returns the N-point frequency vector W in radians and the N-point complex frequency response vector H given numerator and denominator coefficients in vectors B and A. The frequency response is evaluated at N points equally spaced around the upper half of the unit circle. If N isn't specified, it defaults to 512.

```
>> omega=(0:400)*pi/400; b=[1,1,1,1,1]/5;
>> H=freqz(b,1,omega);
>> subplot(211); plot(omega/pi,abs(H))
>> subplot(212); plot(omega/pi,angle(H))
```

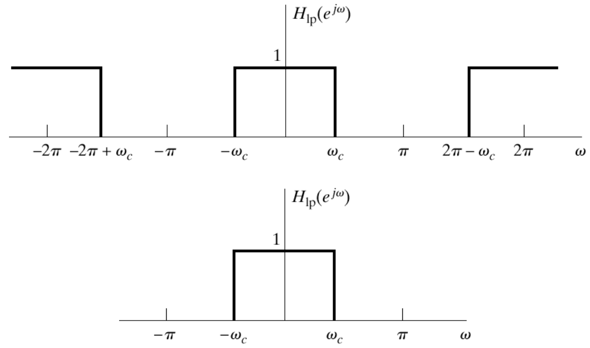
Plotting the Frequency Response

$$H(e^{j\omega}) = \frac{1}{5}\frac{\sin[\omega 5/2]}{\sin(\omega/2)}e^{-j\omega 2}$$

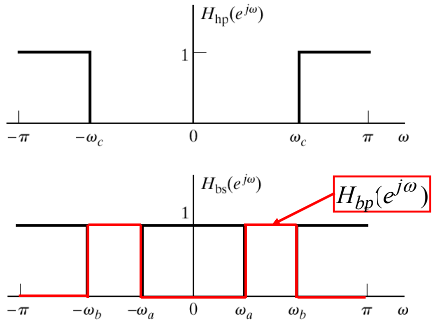
$M = 4$



Ideal Lowpass Filter



Frequency Selective Filters



Suddenly Applied Complex Exponential to a Causal System

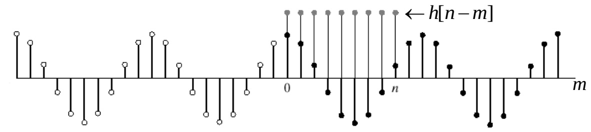
$$x[n] = e^{j\omega n}u[n] \mapsto y[n] = \begin{cases} 0 & n < 0 \\ \left(\sum_{k=0}^{n} h[k]e^{-j\omega k}\right)e^{j\omega n} & n \geq 0 \end{cases}$$

$$y[n] = \left(\sum_{k=0}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

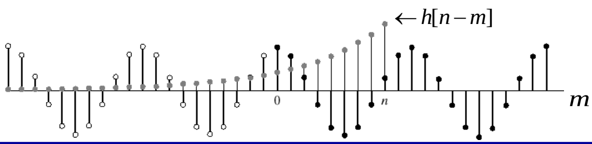
$$y[n] = H(e^{j\omega})e^{j\omega n} - \left(\sum_{m=-1}^{-\infty} h[n-m]e^{-j\omega(n-m)}\right)e^{j\omega n}$$

$$y[n] = H(e^{j\omega})e^{j\omega n} - \left(\sum_{m=-\infty}^{-1} h[n-m]e^{j\omega m}\right)$$

Sinusoidal Steady State



$$y[n] = H(e^{j\omega})e^{j\omega n} - \left(\cancelto{0}{\sum_{m=-\infty}^{-1} h[n-m]e^{j\omega m}} \right)$$



Response to a Cosine

$$x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

$$y[n] = H(e^{j\omega_0})\frac{A}{2}e^{j\phi}e^{j\omega_0 n} + H(e^{-j\omega_0})\frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

$$y[n] = \left|H(e^{j\omega_0})\right| e^{j\angle H(e^{j\omega_0})} \frac{A}{2}e^{j\phi}e^{j\omega_0 n}$$

$$+ \left|H(e^{j\omega_0})\right| e^{-j\angle H(e^{j\omega_0})} \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

$$\boxed{y[n] = \left|H(e^{j\omega_0})\right| A\cos[\omega_0 n + \phi + \angle H(e^{j\omega_0})]}$$

Discrete-Time Fourier Transform

- Definition:

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}}$$ Direct Transform

$$\boxed{x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega}$$ Inverse Transform

- Existence of the DTFT:

$$\left|X(e^{j\omega})\right| \le \sum_{n=-\infty}^{\infty} |x[n]| \cancelto{1}{\left|e^{-j\omega n}\right|} \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Frequency Response Again

- The frequency response function is now seen to be just the DTFT of the impulse response.

$$\boxed{H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}}$$

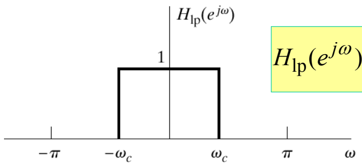
- Note that stable systems have frequency responses.

$$\boxed{\sum_{n=-\infty}^{\infty} |h[n]| < \infty}$$

- Therefore, the impulse response is just the inverse DTFT of the frequency response.

$$\boxed{h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega}$$

Ideal Lowpass Filter



$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_{\text{lp}}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\text{lp}}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$

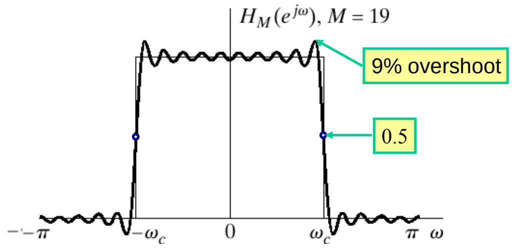
$$= \frac{1}{j2\pi n}\left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{j2\pi n}$$

$$h_{\text{lp}}[n] = \frac{\sin\omega_c n}{\pi n} \qquad -\infty < n < \infty$$

Convergence

- It should be true that

$$\lim_{M\to\infty}\left(\sum_{n=-M}^{M}\frac{\sin\omega_c n}{\pi n}e^{-j\omega n}\right) \to \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$



Example of DTFT

- Consider a signal $x[n] = Ar^n\cos(\omega_0 n + \phi)u[n]$
- To find its DTFT, break it up into complex exponentials

$$x[n] = \tfrac{A}{2}e^{j\phi}r^n e^{j\omega_0 n}u[n] + \tfrac{A}{2}e^{-j\phi}r^n e^{-j\omega_0 n}u[n]$$

- Therefore, the DTFT is

$$X(e^{j\omega}) = \frac{\frac{A}{2}e^{j\phi}}{1 - re^{j\omega_0}e^{-j\omega}} + \frac{\frac{A}{2}e^{-j\phi}}{1 - re^{-j\omega_0}e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{A\cos\phi - Ar\cos(\omega_0 - \phi)e^{-j\omega}}{1 - 2r\cos\omega_0 e^{-j\omega} + r^2 e^{-j2\omega}}$$

More Example of DTFT

- Consider the real exponential signal

$$x[n] = a^n u[n]$$

- Its DTFT is obtained as follows:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty}\left(ae^{-j\omega}\right)^n = \frac{1}{1 - ae^{-j\omega}}$$

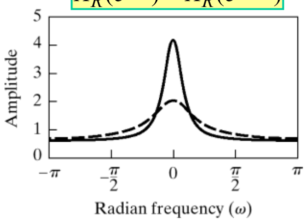
$$X(e^{j\omega}) = \frac{1 - ae^{j\omega}}{(1 - ae^{-j\omega})(1 - ae^{j\omega})}$$

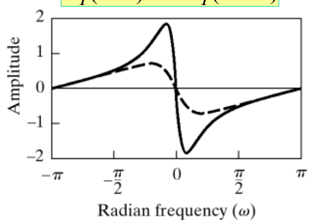
$$= \underbrace{\frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega}}_{X_R(e^{j\omega})} + j\underbrace{\frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega}}_{X_I(e^{j\omega})}$$

Real and Imaginary Parts

$$X(e^{j\omega}) = \underbrace{\frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega}}_{X_R(e^{j\omega})} + j\underbrace{\frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega}}_{X_I(e^{j\omega})}$$

$$\boxed{X_R(e^{j\omega}) = X_R(e^{-j\omega})} \qquad \boxed{X_I(e^{j\omega}) = -X_I(e^{-j\omega})}$$





Magnitude and Angle Form

$$\boxed{x[n] = a^n u[n] \quad \Leftrightarrow \quad X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}}$$

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\angle X(e^{j\omega})}$$

$$\left|X(e^{j\omega})\right|^2 = X(e^{j\omega})X^*(e^{j\omega}) = \frac{1}{(1 - ae^{-j\omega})(1 - ae^{j\omega})}$$

$$= \frac{1}{1 + a^2 - 2a\cos\omega}$$

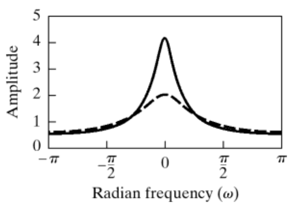
$$\angle X(e^{j\omega}) = \arctan\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right)$$

Magnitude and Angle Plots

$$\left|X(e^{j\omega})\right| = \frac{1}{(1 + a^2 - 2a\cos\omega)^{1/2}}$$

$$\angle X(e^{j\omega}) = \arctan\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right)$$

$$\boxed{\left|X(e^{j\omega})\right| = \left|X(e^{-j\omega})\right|} \qquad \boxed{\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})}$$



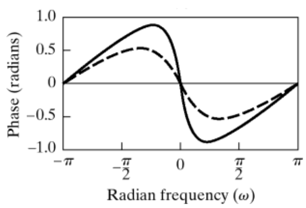


TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Frequency Response of a DE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k Y(e^{j\omega}) e^{-j\omega k} = \sum_{k=0}^{M} b_k X(e^{j\omega}) e^{-j\omega k}$$

$$\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right) Y(e^{j\omega}) = \left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right) X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right)}{\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right)}$$

Using the DTFT

- The DTFT provides a “frequency-domain” representation that is invaluable for thinking about and solving DSP problems.
- To use it effectively you must
  - know the Fourier transforms of certain important signals
  - know its properties and certain key theorems
  - be able to combine time-domain and frequency domain methods appropriately

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin \omega_p(n+1)}{\sin \omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin \omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$

We worked this one out.

And this one too.

The Unit Impulse Function

- The continuous-variable impulse is defined by the following properties:

$$\delta(\omega) = 0 \ \text{ for } \ \omega \neq 0 \qquad \text{highly concentrated}$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(\omega) d\omega = 1 \ \text{ for } \varepsilon > 0 \qquad \text{area concentrated}$$

$$X(e^{j\omega})\delta(\omega) = X(e^{j0})\delta(\omega) \qquad \text{sampling property}$$

$$\delta(\omega) * X(e^{j\omega}) = X(e^{j\omega}) \quad \text{replicating property}$$

DTFT of a Constant Signal

- Assume a “periodic impulse train” for the DTFT

$$\boxed{x[n]=1 \Leftrightarrow X(e^{j\omega})=\sum_{r=-\infty}^{\infty} 2\pi\delta(\omega+2\pi r)}$$

- Find the corresponding sequence $x[n]$

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{r=-\infty}^{\infty} 2\pi\delta(\omega+2\pi r)e^{j\omega n}d\omega$$

$$=\frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi}\cancel{2\pi}\delta(\omega)e^{j\omega n}d\omega = \int_{-\pi}^{\pi}\delta(\omega)e^{j0n}d\omega=1$$

Very Useful DTFT Pairs

$$x[n]=\delta[n-n_d] \Leftrightarrow X(e^{j\omega})=e^{-j\omega n_d}$$

$$x[n]=a^n u[n] \Leftrightarrow X(e^{j\omega})=\frac{1}{1-ae^{-j\omega}}$$

$$x[n]=1 \Leftrightarrow X(e^{j\omega})=\sum_{r=-\infty}^{\infty} 2\pi\delta(\omega+2\pi r)$$

$$x[n]=\frac{\sin\omega_c n}{\pi n} \Leftrightarrow X(e^{j\omega})=\begin{cases}0 & |\omega|<\omega_c\\ 1 & |\omega|>\omega_c\end{cases}$$

$$x[n]=\begin{cases}1 & 0\le n\le M\\ 0 & \text{otherwise}\end{cases} \Leftrightarrow X(e^{j\omega})=\frac{\sin[(M+1)\omega/2]}{\sin(\omega/2)}e^{-j\omega M}$$