ECE4270
Fundamentals of DSP
Lecture 5

The Discrete-Time Fourier Transform and Random Processes

School of ECE

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Overview of Lecture 5

- Examples of DTFT
- Symmetry properties of the DTFT
- Fourier Transform Pairs
- Fourier Transform Theorems
- The Fourier-domain convolution theorem
- Examples of usage of the DTFT

Discrete-Time Fourier Transform

- Definition:

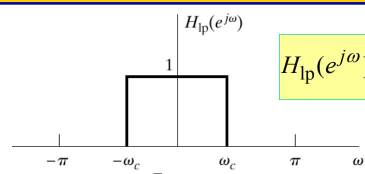
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$ Direct Transform

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$ Inverse Transform

- Existence of the DTFT:

$$\left|X(e^{j\omega})\right| \le \sum_{n=-\infty}^{\infty} |x[n]|\overbrace{\left|e^{-j\omega n}\right|}^{1} \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Ideal Lowpass Filter



$$H_{lp}(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{lp}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$

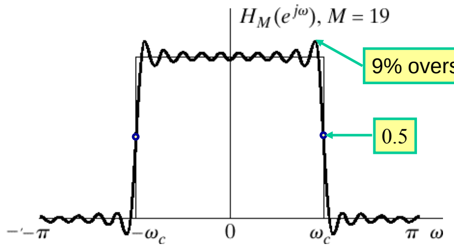
$$= \frac{1}{j2\pi n}\left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{j2\pi n}$$

$$h_{lp}[n] = \frac{\sin\omega_c n}{\pi n} \qquad -\infty < n < \infty$$

Convergence

- It should be true that

$$\lim_{M\to\infty}\left(\sum_{n=-M}^{M}\frac{\sin\omega_c n}{\pi n}e^{-j\omega n}\right)\to\begin{cases}1 & |\omega|<\omega_c\\ 0 & \omega_c<|\omega|\le\pi\end{cases}$$



Example of DTFT

- Consider the real exponential signal

$$\boxed{x[n]=a^n u[n]}$$

- Its DTFT is obtained as follows:

$$X(e^{j\omega})=\sum_{n=0}^{\infty}a^n e^{-j\omega n}=\sum_{n=0}^{\infty}\left(ae^{-j\omega}\right)^n=\frac{1}{1-ae^{-j\omega}}$$

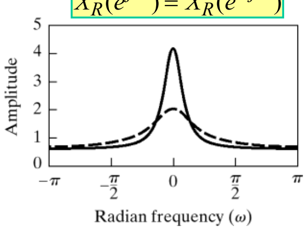
$$X(e^{j\omega})=\frac{1-ae^{j\omega}}{(1-ae^{-j\omega})(1-ae^{j\omega})}$$

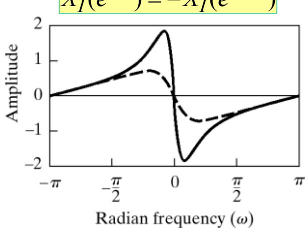
$$=\underbrace{\frac{1-a\cos\omega}{1+a^2-2a\cos\omega}}_{X_R(e^{j\omega})}+j\underbrace{\frac{-a\sin\omega}{1+a^2-2a\cos\omega}}_{X_I(e^{j\omega})}$$

Real and Imaginary Parts

$$X(e^{j\omega})=\underbrace{\frac{1-a\cos\omega}{1+a^2-2a\cos\omega}}_{X_R(e^{j\omega})}+j\underbrace{\frac{-a\sin\omega}{1+a^2-2a\cos\omega}}_{X_I(e^{j\omega})}$$

$$\boxed{X_R(e^{j\omega})=X_R(e^{-j\omega})}\qquad\boxed{X_I(e^{j\omega})=-X_I(e^{-j\omega})}$$





Magnitude and Angle Form

$$\boxed{x[n]=a^n u[n]\quad\Leftrightarrow\quad X(e^{j\omega})=\frac{1}{1-ae^{-j\omega}}}$$

$$X(e^{j\omega})=\left|X(e^{j\omega})\right|e^{j\angle X(e^{j\omega})}$$

$$\left|X(e^{j\omega})\right|^2=X(e^{j\omega})X^*(e^{j\omega})=\frac{1}{(1-ae^{-j\omega})(1-ae^{j\omega})}$$

$$=\frac{1}{1+a^2-2a\cos\omega}$$

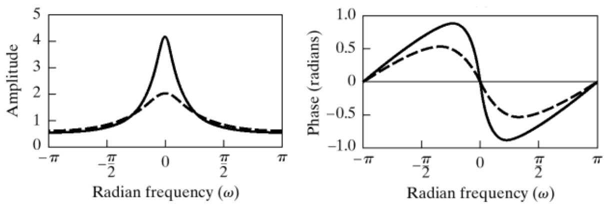
$$\angle X(e^{j\omega})=\arctan\left(\frac{-a\sin\omega}{1-a\cos\omega}\right)$$

Magnitude and Angle Plots

$$|X(e^{j\omega})| = \frac{1}{(1+a^2-2a\cos\omega)^{1/2}}$$

$$\angle X(e^{j\omega}) = \arctan\left(\frac{-a\sin\omega}{1-a\cos\omega}\right)$$

$$\boxed{|X(e^{j\omega})| = |X(e^{-j\omega})|} \qquad \boxed{\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})}$$



More Example of DTFT

- Consider a signal $x[n] = Ar^n\cos(\omega_0 n + \phi)u[n]$
- To find its DTFT, break it up into complex exponentials

$$x[n] = \frac{A}{2}e^{j\phi}r^n e^{j\omega_0 n}u[n] + \frac{A}{2}e^{-j\phi}r^n e^{-j\omega_0 n}u[n]$$

- Therefore, the DTFT is

$$X(e^{j\omega}) = \frac{\frac{A}{2}e^{j\phi}}{1-re^{j\omega_0}e^{-j\omega}} + \frac{\frac{A}{2}e^{-j\phi}}{1-re^{-j\omega_0}e^{-j\omega}}$$

$$\boxed{X(e^{j\omega}) = \frac{A\cos\phi - Ar\cos(\omega_0-\phi)e^{-j\omega}}{1-2r\cos\omega_0 e^{-j\omega} + r^2 e^{-j2\omega}}}$$

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\|X(e^{j\omega})\| = \|X(e^{-j\omega})\|$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

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Using the DTFT

- The DTFT provides a “frequency-domain” representation that is invaluable for thinking about and solving DSP problems.
- To use it effectively you must
  - know the Fourier transforms of certain important signals
  - know its properties and certain key theorems
  - be able to combine time-domain and frequency domain methods appropriately

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$

We worked this one out.

And this one too.

The Unit Impulse Function

- The continuous-variable impulse is defined by the following properties:

$$\delta(\omega) = 0 \quad \text{for} \quad \omega \neq 0 \qquad \text{highly concentrated}$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(\omega)d\omega = 1 \quad \text{for } \varepsilon > 0 \qquad \text{area concentrated}$$

$$X(e^{j\omega})\delta(\omega) = X(e^{j0})\delta(\omega) \quad \text{sampling property}$$

$$\delta(\omega) * X(e^{j\omega}) = X(e^{j\omega}) \quad \text{replicating property}$$

DTFT of a Constant Signal

- Assume a “periodic impulse train” for the DTFT

$$\boxed{x[n] = 1 \Leftrightarrow X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega+2\pi r)}$$

- Find the corresponding sequence $x[n]$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega+2\pi r)e^{j\omega n}d\omega$$

$$= \frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi} \cancel{2\pi}\delta(\omega)e^{j\omega n}d\omega = \int_{-\pi}^{\pi} \delta(\omega)e^{j0n}d\omega = 1$$

Very Useful DTFT Pairs

$$x[n] = \delta[n-n_d] \Leftrightarrow X(e^{j\omega}) = e^{-j\omega n_d}$$

$$x[n] = a^n u[n] \Leftrightarrow X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

$$x[n] = 1 \Leftrightarrow X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

$$x[n] = \frac{\sin \omega_c n}{\pi n} \Leftrightarrow X(e^{j\omega}) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & |\omega| > \omega_c \end{cases}$$

$$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow X(e^{j\omega}) = \frac{\sin[(M+1)\omega/2]}{\sin(\omega/2)} e^{-j\omega M}$$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n-n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega}) d\omega$

DTFT of Sinusoids

- Recall that

$$\boxed{x[n] = 1 \Leftrightarrow X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)}$$

- Also note that

$$x[n]e^{j\omega_0 n} \Leftrightarrow X(e^{j(\omega-\omega_0)})$$

- Therefore

$$\boxed{e^{j\omega_0 n} \Leftrightarrow \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi r)}$$

$$\boxed{\cos \omega_0 n \Leftrightarrow \sum_{r=-\infty}^{\infty} \pi\delta(\omega + \omega_0 + 2\pi r) + \pi\delta(\omega - \omega_0 + 2\pi r)}$$

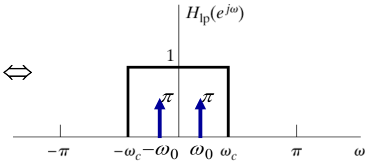
Convolution Theorem

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \Leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})}$$

$x[n]$	LTI System	$y[n]$
$\delta[n]$		$h[n]$
$e^{j\omega n}$		$H(e^{j\omega})e^{j\omega n}$
$X(e^{j\omega})$		$X(e^{j\omega})H(e^{j\omega})$

Example 1

$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c n}{\pi n} \quad \Leftrightarrow$$



- Find the output when the input is

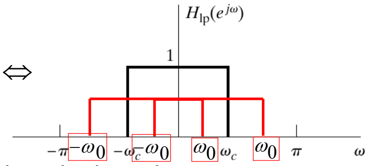
$x[n] = \cos \omega_0 n \Leftrightarrow$

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \pi\delta(\omega + \omega_0 + 2\pi r) + \pi\delta(\omega - \omega_0 + 2\pi r)$$

$$y[n] = \cos \omega_0 n \quad \text{if} \ \ \omega_0 < \omega_c$$

Example 2

$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c n}{\pi n} \quad \Leftrightarrow$$



- Find the output when the input is

$$x[n] = \frac{\sin(\omega_0 n)}{2\pi n}$$

$$\boxed{y[n] = \frac{\sin(\omega_0 n)}{2\pi n} \ \text{if} \ \omega_c \lessgtr \omega_0}$$

Frequency Response of a DE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k Y(e^{j\omega}) e^{-j\omega k} = \sum_{k=0}^{M} b_k X(e^{j\omega}) e^{-j\omega k}$$

$$\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right) Y(e^{j\omega}) = \left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right) X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right)}{\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right)}$$

Example 3

- Suppose that the difference equation is

$$y[n] = y[n-1] - .9y[n-2] + x[n] + x[n-1]$$

- The frequency response is

$$H(e^{j\omega}) = \frac{1 + e^{-j\omega}}{1 - e^{-j\omega} + .9e^{-j\omega 2}}$$

- This system is implemented in MATLAB by

```
>> y=filter([1,1],[1,-1,.9],x)
```

We can compute its frequency response by

```
>> omega=(0:500)*pi/500;
>> H=freqz ([1,1],[1,-1,.9],omega);
```