ECE4270
Fundamentals of DSP

Lectures 6

Random Processes

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Overview of Lectures 6

- Fourier Transform Theorems (Lecture 5)
- The Fourier-domain convolution theorem (Lecture 5)
- Examples of usage of the DTFT (Lecture 5)
- Frequency Response of DE (Lecture 5)
- Random process
  - Probability distributions
  - Averages: Mean, variance, correlation
- Stationary random processes
- Time averages and ergodic random processes
- The Bernoulli random process

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d]$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

DTFT of Sinusoids

- Recall that

$$x[n]=1 \Leftrightarrow X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega+2\pi r)$$

- Also note that

$$x[n]e^{j\omega_0 n} \Leftrightarrow X(e^{j(\omega-\omega_0)})$$

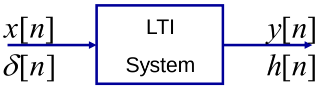
- Therefore

$$e^{j\omega_0 n} \Leftrightarrow \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi r)$$

$$\cos\omega_0 n \Leftrightarrow \sum_{r=-\infty}^{\infty} \pi\delta(\omega+\omega_0+2\pi r)+\pi\delta(\omega-\omega_0+2\pi r)$$

Convolution Theorem

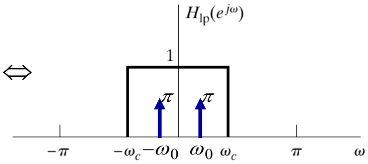
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \Leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$



Input	Output
$x[n]$	$y[n]$
$\delta[n]$	$h[n]$
$e^{j\omega n}$	$H(e^{j\omega})e^{j\omega n}$
$X(e^{j\omega})$	$X(e^{j\omega})H(e^{j\omega})$

Example 1

$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n} \quad \Leftrightarrow$$



- Find the output when the input is

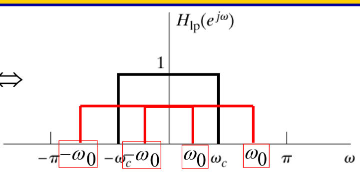
$$x[n] = \cos \omega_0 n \Leftrightarrow$$

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \pi\delta(\omega + \omega_0 + 2\pi r) + \pi\delta(\omega - \omega_0 + 2\pi r)$$

$$y[n] = \cos \omega_0 n \quad \text{if } \omega_0 < \omega_c$$

Example 2

$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n} \quad \Leftrightarrow$$



- Find the output when the input is

$$x[n] = \frac{\sin(\omega_0 n)}{2\pi n}$$

$$y[n] = \frac{\sin(\omega_0 n)}{2\pi n} \quad \text{if } \omega_c > \omega_0$$

Frequency Response of a DE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k Y(e^{j\omega})e^{-j\omega k} = \sum_{k=0}^{M} b_k X(e^{j\omega})e^{-j\omega k}$$

$$\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right) Y(e^{j\omega}) = \left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right) X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right)}{\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right)}$$

Example 3

- Suppose that the difference equation is

$$y[n] = y[n-1] - .9y[n-2] + x[n] + x[n-1]$$

- The frequency response is

$$H(e^{j\omega}) = \frac{1 + e^{-j\omega}}{1 - e^{-j\omega} + .9e^{-j\omega 2}}$$

- This system is implemented in MATLAB by

```
>> y=filter([1,1],[1,-1,.9],x)
```

We can compute its frequency response by

```
>> omega=(0:500)*pi/500;
>> H=freqz ([1,1],[1,-1,.9],omega);
```


What is a random signal?

- Many signals vary in complicated patterns that cannot easily be described by simple equations.
  - It is often convenient and useful to consider such signals as being created by some sort of random mechanism.
  - Many such signals are considered to be "noise", although this is not always the case.
- The mathematical representation of "random signals" involves the concept of a *random process*.

Random Process

- A random process is an indexed set of random variables $\{\mathbf{x}_n\}$, each of which is characterized by a probability distribution (or density)

$$P_{\mathbf{x}_n}(x_n, n) = \text{Prob}\{\mathbf{x}_n \le x_n\} = \int_{-\infty}^{x_n} p_{\mathbf{x}_n}(x, n)dx$$

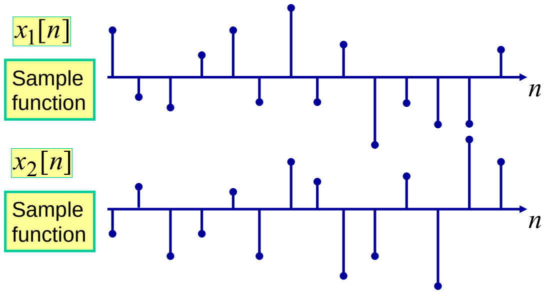
$$p_{\mathbf{x}_n}(x_n, n) = \frac{\partial P_{\mathbf{x}_n}(x_n, n)}{\partial x_n} \qquad -\infty < n < \infty$$

and the collection of random variables is characterized by a set of joint probability distributions such as (for all *n* and *m)*,

$$P_{\mathbf{x}_n, \mathbf{x}_m}(x_n, n, x_m, m) = \text{Prob}\{\mathbf{x}_n \le x_n \text{ and } \mathbf{x}_m \le x_m\}$$

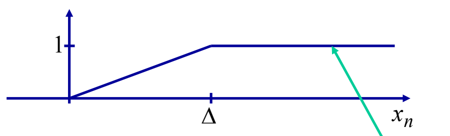
Ensemble of Sample Functions

- We imagine that there are an infinite set of possible sequences where the value at n is governed by a probability law. We call this set an ensemble.

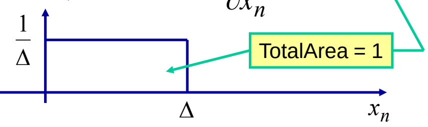


Uniform Distribution

$$P_{\mathbf{x}_n}(x_n,n) = \text{Prob}\{\mathbf{x}_n \le x_n\}$$

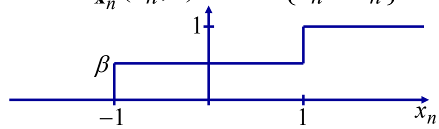


$$p_{\mathbf{x}_n}(x_n,n) = \frac{\partial P_{\mathbf{x}_n}(x_n,n)}{\partial x_n}$$

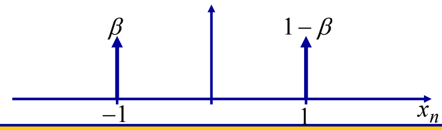


Bernoulli Distribution

$$P_{\mathbf{x}_n}(x_n,n) = \text{Prob}\{\mathbf{x}_n \le x_n\}$$



$$p_{\mathbf{x}_n}(x_n,n) = \frac{\partial P_{\mathbf{x}_n}(x_n,n)}{\partial x_n}$$



Averages of Random Processes

- Mean (expected value) of a random process

$$m_{\mathbf{x}_n} = E\{\mathbf{x}_n\} = \int_{-\infty}^{\infty} x p_{\mathbf{x}_n}(x,n)dx$$

- Expected value of a function of a random process

$$E\{g(\mathbf{x}_n)\} = \int_{-\infty}^{\infty} g(x) p_{\mathbf{x}_n}(x,n)dx$$

- In general such averages will depend upon *n*. However, for a *stationary random process*, all the first-order averages are the same; e.g.,

$$m_{\mathbf{x}_n} = m_x \quad \text{for all } n$$

More Averages

- Mean-squared (average power)

$$E\{\mathbf{x}_n \mathbf{x}_n^*\} = E\{|\mathbf{x}_n|^2\} = \int_{-\infty}^{\infty} x^2 p_{\mathbf{x}_n}(x,n)dx$$

- Variance

$$\text{var}[\mathbf{x}_n] = E\{(\mathbf{x}_n - m_{\mathbf{x}_n})(\mathbf{x}_n - m_{\mathbf{x}_n})^*\} = \sigma_{\mathbf{x}_n}^2$$

$$\text{var}[\mathbf{x}_n] = E\{\mathbf{x}_n \mathbf{x}_n^*\} - |m_{\mathbf{x}_n}|^2 = \sigma_{\mathbf{x}_n}^2$$

$$\text{var}[\mathbf{x}_n] = (\text{mean}-\text{square}) - (\text{mean})^2 = \sigma_{\mathbf{x}_n}^2$$

Joint Averages of Two R.V.s

- Expected value of a function of two random processes.

$$E\{g(\mathbf{x}_n,\mathbf{y}_m)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)p_{\mathbf{x}_n,\mathbf{y}_m}(x,n,y,m)dxdy$$

- Two random processes are *uncorrelated* if

$$E\{\mathbf{x}_n\mathbf{y}_m\}=E\{\mathbf{x}_n\}E\{\mathbf{y}_m\}$$

- Statistical independence implies

$$p_{\mathbf{x}_n,\mathbf{y}_m}(x,n,y,m)=p_{\mathbf{x}_n}(x,n)p_{\mathbf{y}_m}(y,m)$$

- Independent random processes are also uncorrelated.

Correlation Functions

- Autocorrelation function

$$\phi_{xx}[n,m]=E\{\mathbf{x}_n\mathbf{x}_m^*\}$$

- Autocovariance function

$$\gamma_{xx}[n,m]=E\{(\mathbf{x}_n-m_{\mathbf{x}_n})(\mathbf{x}_m-m_{\mathbf{x}_m})^*\}$$

- Crosscorrelation function

$$\phi_{xy}[n,m]=E\{\mathbf{x}_n\mathbf{y}_m^*\}$$

- Crosscovariance function

$$\gamma_{xy}[n,m]=E\{(\mathbf{x}_n-m_{\mathbf{x}_n})(\mathbf{y}_m-m_{\mathbf{y}_m})^*\}$$

Stationary Random Processes

- The probability distributions do not change with time.

$$p_{\mathbf{x}_{n+k}}(x_n,n)=p_{\mathbf{x}_n}(x_n,n)$$

$$p_{\mathbf{x}_{n+k},\mathbf{x}_{m+k}}(x_n,n,x_m,m)=p_{\mathbf{x}_n,\mathbf{x}_m}(x_n,n,x_m,m)$$

- Thus, mean and variance are constant

$$m_x=E\{\mathbf{x}_n\}$$

$$\sigma_x^2=E\{(\mathbf{x}_n-m_x)(\mathbf{x}_n-m_x)^*\}$$

- And the autocorrelation is a one-dimensional function of the time difference.

$$\phi_{xx}[n+m,n]=\phi_{xx}[m]=E\{\mathbf{x}_{n+m}\mathbf{x}_n^*\}$$

Time Averages

- Time-averages of a random process are random variables themselves.

$$\langle\mathbf{x}_n\rangle=\lim_{L\to\infty}\frac{1}{2L+1}\sum_{n=-L}^{L}\mathbf{x}_n$$

$$\langle\mathbf{x}_{n+m}\mathbf{x}_n^*\rangle=\lim_{L\to\infty}\frac{1}{2L+1}\sum_{n=-L}^{L}\mathbf{x}_{n+m}\mathbf{x}_n^*$$

- Time averages of a single sample function

$$\langle x[n]\rangle=\lim_{L\to\infty}\frac{1}{2L+1}\sum_{n=-L}^{L}x[n]$$

$$\langle x[n+m]x^*[n]\rangle=\lim_{L\to\infty}\frac{1}{2L+1}\sum_{n=-L}^{L}x[n+m]x^*[n]$$

Ergodic Random Processes

- Time-averages are equal to probability averages

$$\langle \mathbf{x}_n \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \mathbf{x}_n = E\{\mathbf{x}_n\} = m_x$$

$$\langle \mathbf{x}_{n+m}\mathbf{x}_n^* \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \mathbf{x}_{n+m}\mathbf{x}_n^*$$

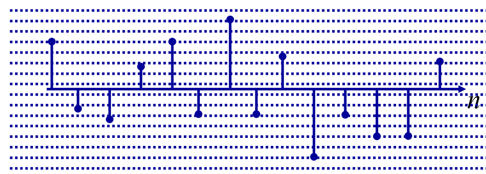
$$= E\{\mathbf{x}_{n+m}\mathbf{x}_n^*\} = \phi_{xx}[m]$$

- Estimates from a single sample function

$$\hat{m}_x = \frac{1}{L}\sum_{n=0}^{L-1} x[n] \qquad \hat{\phi}_{xx}[m] = \frac{1}{L}\sum_{n=0}^{L-1} x[n+m]x^*[n]$$

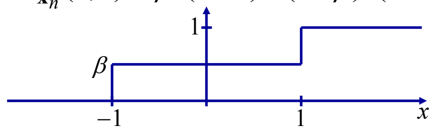
Histogram

- A histogram shows counts of samples that fall in certain "bins". If the boundaries of the bins are close together and we use a sample function with many samples, the histogram provides a good estimate of the probability density function of an (assumed) stationary random process.

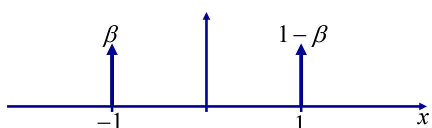


Bernoulli Distribution

$$P_{\mathbf{x}_n}(x,n) = \beta u(x+1) + (1-\beta)u(x-1)$$



$$p_{\mathbf{x}_n}(x,n) = \frac{\partial P_{\mathbf{x}_n}(x,n)}{\partial x} = \beta\delta(x+1) + (1-\beta)\delta(x-1)$$



Bernoulli Random Process

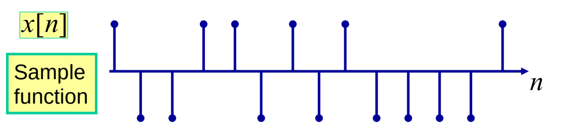
- Suppose that the signal takes on only two different values +1 or -1 with equal probability.

$$P_{\mathbf{x}_n}(x_n,n) = 0.5u(x_n+1) + 0.5u(x_n-1)$$

$$p_{\mathbf{x}_n}(x_n,n) = 0.5\delta(x_n+1) + 0.5\delta(x_n-1)$$

Furthermore, assume that the outcome at time n is independent of all other outcomes.

$$P_{\mathbf{x}_n,\mathbf{x}_m}(x_n,n,x_m,m) = P_{\mathbf{x}_n}(x_n,n)P_{\mathbf{x}_m}(x_m,m)$$



Bernoulli Process (cont.)

- Mean: $m_x = \int_{-\infty}^{\infty} x[0.5\delta(x+1)+0.5\delta(x-1)]dx$

$$m_x = \int_{-\infty}^{\infty} 0.5x\delta(x+1)dx + \int_{-\infty}^{\infty} 0.5x\delta(x-1)dx$$

$$m_x = -0.5+0.5=0$$

- Variance:

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-m_x)^2[0.5\delta(x+1)+0.5\delta(x-1)]dx$$

$$\sigma_x^2 = 0.5+0.5=1$$

- Autocorrelation: ($\{\mathbf{x}_n\}$ are assumed independent)

$$\phi_{xx}[m]=\sigma_x^2\delta[m]=\delta[m]$$

MATLAB Bernoulli Simulation

- MATLAB's rand( ) function is useful for such simulations.

```
>> d = rand(1,N); %uniform dist. Between 0 & 1
>> k = find(x>.5); %find +1s
>> x = -ones(1,N);          %make vector of all -1s
>> x(k) = ones(1,length(k));      %insert +1s
>> subplot(211); han=stem(0:Nplt-1,x(1:Nplt));
>> set(han,'markersize',3);
>> subplot(212);  hist(x,Nbins); hold on
>> stem([-1,1],N*[.5,.5],'r*'); %add theoretical values
```


Bernoulli Random Process