ECE4270
Fundamentals of DSP

Lectures 7

Random Processes and LTI Filtering

School of ECE

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Overview of Lectures 7

- Random process (from Lecture 6)
  - Probability distributions
  - Averages: Mean, variance, correlation
- Stationary random processes
- The Bernoulli and Uniform random process
- Linear systems with random inputs
- Power Density Spectrum

Random Process

- A random process is an indexed set of random variables $\{\mathbf{x}_n\}$, each of which is characterized by a probability distribution (or density)

$$P_{\mathbf{x}_n}(x_n,n) = \text{Prob}\{\mathbf{x}_n \le x_n\} = \int_{-\infty}^{x_n} p_{\mathbf{x}_n}(x,n)dx$$

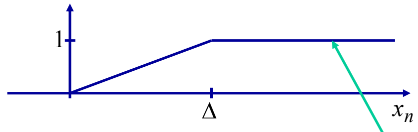
$$p_{\mathbf{x}_n}(x_n,n) = \frac{\partial P_{\mathbf{x}_n}(x_n,n)}{\partial x_n} \qquad -\infty < n < \infty$$

and the collection of random variables is characterized by a set of joint probability distributions such as (for all *n* and *m),*

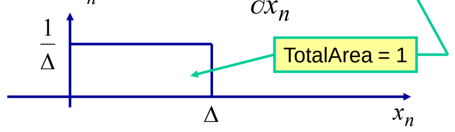
$$P_{\mathbf{x}_n,\mathbf{x}_m}(x_n,n,x_m,m) = \text{Prob}\{\mathbf{x}_n \le x_n \text{ and } \mathbf{x}_m \le x_m\}$$

Uniform Distribution

$$P_{\mathbf{x}_n}(x_n,n) = \text{Prob}\{\mathbf{x}_n \le x_n\}$$

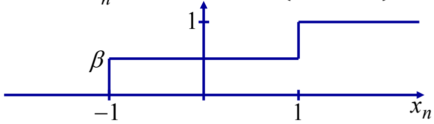


$$p_{\mathbf{x}_n}(x_n,n) = \frac{\partial P_{\mathbf{x}_n}(x_n,n)}{\partial x_n}$$

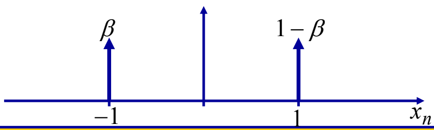


Bernoulli Distribution

$$P_{\mathbf{x}_n}(x_n,n) = \text{Prob}\{\mathbf{x}_n \le x_n\}$$



$$p_{\mathbf{x}_n}(x_n,n) = \frac{\partial P_{\mathbf{x}_n}(x_n,n)}{\partial x_n}$$



Averages of Random Processes

- Mean (expected value) of a random process

$$m_{\mathbf{x}_n} = E\{\mathbf{x}_n\} = \int_{-\infty}^{\infty} x p_{\mathbf{x}_n}(x,n)dx$$

- Expected value of a function of a random process

$$E\{g(\mathbf{x}_n)\} = \int_{-\infty}^{\infty} g(x) p_{\mathbf{x}_n}(x,n)dx$$

- In general such averages will depend upon *n*. However, for a *stationary random process*, all the first-order averages are the same; e.g.,

$$m_{\mathbf{x}_n} = m_x \quad \text{for all } n$$

More Averages

- Mean-squared (average power)

$$E\{\mathbf{x}_n \mathbf{x}_n^*\} = E\{|\mathbf{x}_n|^2\} = \int_{-\infty}^{\infty} x^2 p_{\mathbf{x}_n}(x,n)dx$$

- Variance

$$\text{var}[\mathbf{x}_n] = E\{(\mathbf{x}_n - m_{\mathbf{x}_n})(\mathbf{x}_n - m_{\mathbf{x}_n})^*\} = \sigma_{\mathbf{x}_n}^2$$

$$\text{var}[\mathbf{x}_n] = E\{\mathbf{x}_n \mathbf{x}_n^*\} - |m_{\mathbf{x}_n}|^2 = \sigma_{\mathbf{x}_n}^2$$

$$\text{var}[\mathbf{x}_n] = (\text{mean} - \text{square}) - (\text{mean})^2 = \sigma_{\mathbf{x}_n}^2$$

Joint Averages of Two R.V.s

- Expected value of a function of two random processes.

$$E\{g(\mathbf{x}_n,\mathbf{y}_m)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) p_{\mathbf{x}_n,\mathbf{y}_m}(x,n,y,m)dxdy$$

- Two random processes are *uncorrelated* if

$$E\{\mathbf{x}_n \mathbf{y}_m\} = E\{\mathbf{x}_n\}E\{\mathbf{y}_m\}$$

- Statistical independence implies

$$p_{\mathbf{x}_n,\mathbf{y}_m}(x,n,y,m) = p_{\mathbf{x}_n}(x,n) p_{\mathbf{y}_m}(y,m)$$

- Independent random processes are also uncorrelated.

Correlation Functions

- Autocorrelation function

$$\phi_{xx}[n,m] = E\left\{\mathbf{x}_n \mathbf{x}_m^*\right\}$$

- Autocovariance function

$$\gamma_{xx}[n,m] = E\left\{(\mathbf{x}_n - m_{\mathbf{x}_n})(\mathbf{x}_m - m_{\mathbf{x}_m})^*\right\}$$

- Crosscorrelation function

$$\phi_{xy}[n,m] = E\left\{\mathbf{x}_n \mathbf{y}_m^*\right\}$$

- Crosscovariance function

$$\gamma_{xy}[n,m] = E\left\{(\mathbf{x}_n - m_{\mathbf{x}_n})(\mathbf{y}_m - m_{\mathbf{y}_m})^*\right\}$$

Stationary Random Processes

- The probability distributions do not change with time. $p_{\mathbf{x}_{n+k}}(x_n,n) = p_{\mathbf{x}_n}(x_n,n)$

$$p_{\mathbf{x}_{n+k},\mathbf{x}_{m+k}}(x_n,n,x_m,m) = p_{\mathbf{x}_n,\mathbf{x}_m}(x_n,n,x_m,m)$$

- Thus, mean and variance are constant

$$m_x = E\left\{\mathbf{x}_n\right\}$$

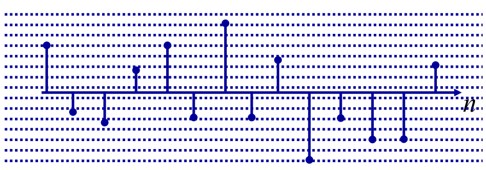
$$\sigma_x^2 = E\left\{(\mathbf{x}_n - m_x)(\mathbf{x}_n - m_x)^*\right\}$$

- And the autocorrelation is a one-dimensional function of the time difference.

$$\phi_{xx}[n+m,n] = \phi_{xx}[m] = E\left\{\mathbf{x}_{n+m}\mathbf{x}_n^*\right\}$$

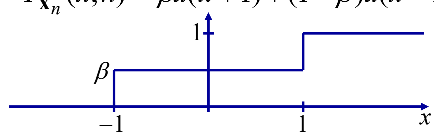
Histogram

- A histogram shows counts of samples that fall in certain “bins”. If the boundaries of the bins are close together and we use a sample function with many samples, the histogram provides a good estimate of the probability density function of an (assumed) stationary random process.

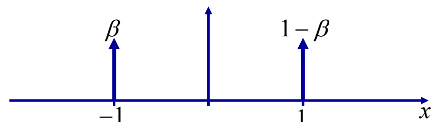


Bernoulli Distribution

$$P_{\mathbf{x}_n}(x,n) = \beta u(x+1) + (1-\beta)u(x-1)$$



$$p_{\mathbf{x}_n}(x,n) = \frac{\partial P_{\mathbf{x}_n}(x,n)}{\partial x} = \beta\delta(x+1) + (1-\beta)\delta(x-1)$$



Bernoulli Random Process

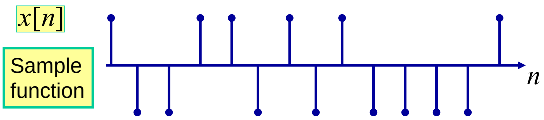
- Suppose that the signal takes on only two different values +1 or -1 with equal probability.

$$P_{\mathbf{x}_n}(x_n,n) = 0.5u(x_n+1) + 0.5u(x_n-1)$$

$$p_{\mathbf{x}_n}(x_n,n) = 0.5\delta(x_n+1) + 0.5\delta(x_n-1)$$

Furthermore, assume that the outcome at time n is independent of all other outcomes.

$$P_{\mathbf{x}_n,\mathbf{x}_m}(x_n,n,x_m,m) = P_{\mathbf{x}_n}(x_n,n)P_{\mathbf{x}_m}(x_m,m)$$



Bernoulli Process (cont.)

- Mean: $m_x = \int_{-\infty}^{\infty} x[0.5\delta(x+1) + 0.5\delta(x-1)]dx$

$$m_x = \int_{-\infty}^{\infty} 0.5x\delta(x+1)dx + \int_{-\infty}^{\infty} 0.5x\delta(x-1)dx$$

$$m_x = -0.5 + 0.5 = 0$$

- Variance:

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-m_x)^2[0.5\delta(x+1) + 0.5\delta(x-1)]dx$$

$$\sigma_x^2 = 0.5 + 0.5 = 1$$

- Autocorrelation: ($\{\mathbf{x}_n\}$ are assumed independent)

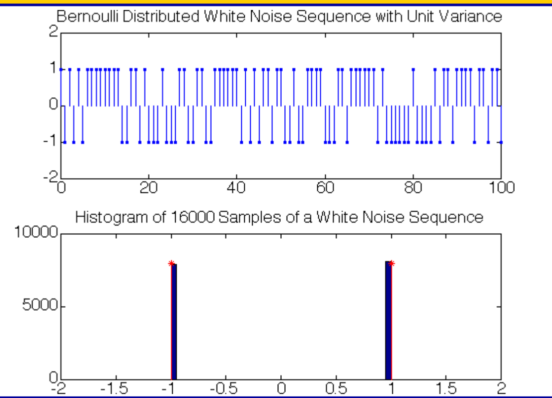
$$\phi_{xx}[m] = \sigma_x^2\delta[m] = \delta[m]$$

MATLAB Bernoulli Simulation

- MATLAB's rand( ) function is useful for such simulations.

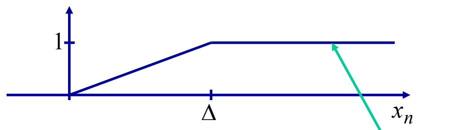
```
>> d = rand(1,N); %uniform dist. Between 0 & 1
>> k = find(x>.5); %find +1s
>> x = -ones(1,N);          %make vector of all -1s
>> x(k) = ones(1,length(k));       %insert +1s
>> subplot(211); han=stem(0:Nplt-1,x(1:Nplt));
>> set(han,'markersize',3);
>> subplot(212);  hist(x,Nbins); hold on
>> stem([-1,1],N*[.5,.5],'r*'); %add theoretical values
```


Bernoulli Random Process

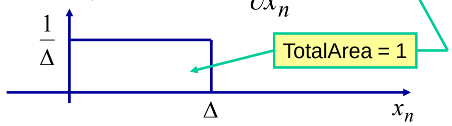


Uniform Distribution

$$P_{\mathbf{x}_n}(x_n,n) = \text{Prob}\{\mathbf{x}_n \le x_n\}$$



$$p_{\mathbf{x}_n}(x_n,n) = \frac{\partial P_{\mathbf{x}_n}(x_n,n)}{\partial x_n}$$



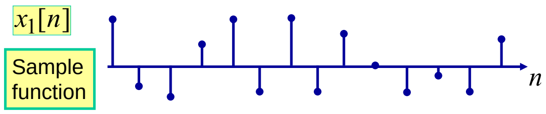
Uniform Random Process

- Suppose that the signal values are uniformly distributed between -1 and +1.

$$p_{\mathbf{x}_n}(x_n,n) = \begin{cases} 0.5 & -1 < x_n < 1 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, assume that the outcome at time n is independent of all other outcomes.

$$P_{\mathbf{x}_n,\mathbf{x}_m}(x_n,n,x_m,m) = P_{\mathbf{x}_n}(x_n,n)P_{\mathbf{x}_m}(x_m,m)$$



Uniform Process (cont.)

- Mean:

$$m_x = \int_{-1}^{1} x(0.5)dx = \left.\frac{x^2}{4}\right]_{-1}^{1}$$

$$m_x = 0.25(1-1) = 0$$

- Variance:

$$\sigma_x^2 = \int_{-1}^{1} (x-m_x)^2(0.5)dx = \left.\frac{x^3}{6}\right]_{-1}^{1}$$

$$\sigma_x^2 = \frac{1}{3}$$

- Autocorrelation: ($\{\mathbf{x}_n\}$ are assumed independent)

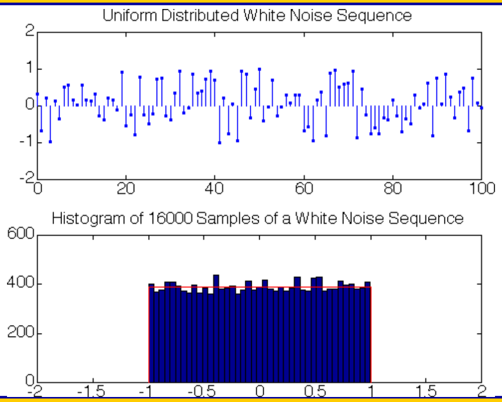
$$\phi_{xx}[m] = \sigma_x^2\delta[m] = \frac{1}{3}\delta[m]$$

MATLAB Uniform Simulation

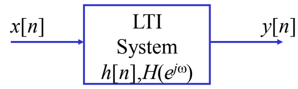
- MATLAB's rand( ) function is useful for such simulations.

```
>> % set Nplt and N
>> x = 2*(rand(1,N)-0.5);          %uniform dist. -1 to 1
>> subplot(211); han=stem(0:Nplt-1,x(1:Nplt));
>> set(han,'markersize',3);
>> subplot(212);  hist(x,Nbins); hold on
>> % add theoretical values
>> plot([-1,-1,1,1],N*[0,.5,.5,0]/Nbins,'r');
```

Uniform Random Process



Linear System with a Random Input

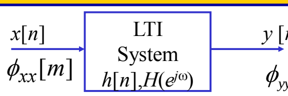


$$m_y = E\left\{\sum_{k=-\infty}^{\infty} x[k]h[n-k]\right\} = \sum_{k=-\infty}^{\infty} E\{x[k]\}h[n-k]$$

$$m_y = E\{x[k]\}\left(\sum_{k=-\infty}^{\infty} h[n-k]\right) = m_x\left(\sum_{k=-\infty}^{\infty} h[k]\right)$$

$$m_y = m_x\left(\sum_{k=-\infty}^{\infty} h[k]\right) = m_x H(e^{j0})$$

Linear System with a Random Input



$$\phi_{yy}[m] = \phi_{xx}[m] * c_{hh}[m]$$

$$\phi_{yy}[m] = E\left\{y[n+m]y^*[n]\right\}$$

Autocorrelation of output

$$= E\left\{\sum_{r=-\infty}^{\infty} x[n+m-r]h[r] \sum_{k=-\infty}^{\infty} x^*[n-k]h^*[k]\right\}$$

$$\phi_{yy}[m] = \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]c_{hh}[\ell]$$

$$c_{hh}[m] = \sum_{k=-\infty}^{\infty} h[m+k]h^*[k]$$

Deterministic autocorrelation of impulse response

Computing Average Power

- Assume a zero mean input whose average power is σ_x^2 and autocorrelation function is $\phi_{xx}[m] = \sigma_x^2\delta[m]$.
- The autocorrelation of the output is

$$\phi_{yy}[m] = \sigma_x^2\delta[m] * c_{hh}[m] = \sigma_x^2 c_{hh}[m]$$

- The average power of the output is easily found from this result as

$$\sigma_y^2 = \phi_{yy}[0] = \sigma_x^2 c_{hh}[0] = \sigma_x^2 \sum_{k=-\infty}^{\infty} h[k]h^*[k]$$

Computing Average Power

- Assume a zero mean input whose average power is σ_x^2 and autocorrelation function is $\phi_{xx}[m]=\sigma_x^2\delta[m]$.
- The autocorrelation of the output is

$$\phi_{yy}[m]=\sigma_x^2\delta[m]*c_{hh}[m]=\sigma_x^2 c_{hh}[m]$$

- The average power of the output is easily found from this result as

$$\sigma_y^2=\phi_{yy}[0]=\sigma_x^2 c_{hh}[0]=\sigma_x^2\sum_{k=-\infty}^{\infty}h[k]h^*[k]$$

Central Limit Theorem

- The probability density of the sum of a large number of independent random variables approaches a Gaussian distribution.

$$p_{\mathbf{y}}(y)=\frac{1}{\sqrt{2\pi}\sigma_y}e^{-(y-m_y)^2/2\sigma_y^2}$$

- Since filters perform a weighted sum of the samples of the input, the output of a digital filter for a random input tends to have a Gaussian distribution.

Properties of the Autocorrelation

- Definition:

$$\phi_{xx}[m]=E\{x[n+m]x^*[n]\}$$

- Average power:

$$\phi_{xx}[0]=E\{|x[n]|^2\}=\text{mean}-\text{square}$$

- Symmetry:

$$\phi_{xx}[-m]=\phi_{xx}^*[m] \qquad \phi_{xx}[-m]=\phi_{xx}[m] \quad \text{if } x \text{ is real}$$

- Shape:

$$|\phi_{xx}[m]|\le\phi_{xx}[0] \qquad \lim_{m\to\infty}\phi_{xx}[m]=|m_x|^2$$

Power Density Spectrum

- The DTFT of a random signal does not exist theoretically. However, the autocorrelation function of the signal does exist, and the power density spectrum of a random signal is the DTFT of the autocorrelation function.

$$\Phi(e^{j\omega})=\sum_{m=-\infty}^{\infty}\phi_{xx}[m]e^{-j\omega m}$$

$$\phi_{xx}[m]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(e^{j\omega})e^{j\omega m}d\omega$$

$$E\{|x|^2\}=\phi_{xx}[0]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(e^{j\omega})d\omega$$

Linear System with a Random Input

$x[n]$ → LTI System $h[n], H(e^{j\omega})$ → $y[n]$

$\phi_{xx}[m]$ → $\phi_{xx}[m] * c_{hh}[m]$

$\Phi_{xx}(e^{j\omega})$ → $\Phi_{xx}(e^{j\omega})C_{hh}(e^{j\omega})$

$$\phi_{yy}[m] = \phi_{xx}[m] * c_{hh}[m]$$

$$c_{hh}[m] = \sum_{k=-\infty}^{\infty} h[m+k]h^*[k] = h[-m] * h^*[m]$$

$$\Phi_{yy}(e^{j\omega}) = \Phi_{xx}(e^{j\omega})C_{hh}(e^{j\omega})$$

$$C_{hh}(e^{j\omega}) = H(e^{-j\omega})H^*(e^{-j\omega}) = \left|H(e^{-j\omega})\right|^2$$

Properties of Power Density

- Real:

$$\Phi^*(e^{j\omega}) = \Phi(e^{j\omega})$$

- Symmetry:

$$\Phi(e^{j\omega}) = \Phi(e^{-j\omega}) \quad \text{if } x[n] \text{ is real}$$

- Positivity:

$$\Phi(e^{j\omega}) \geq 0$$

- Magnitude-squared has same properties:

$$C_{hh}^*(e^{j\omega}) = \left|H(e^{-j\omega})\right|^2 = C_{hh}(e^{j\omega}) \quad \Rightarrow \quad \text{real}$$

$$C_{hh}(e^{-j\omega}) = \left|H(e^{j\omega})\right|^2 = C_{hh}(e^{j\omega}) \quad \text{if } x[n] \text{ is real}$$