ECE4270
Fundamentals of DSP

Lecture 8

Power Spectrum
&
Chapter 3: Z-Transform

School of ECE
Center for Signal and Information Processing
Georgia Institute of Technology

Overview of Lecture 8

- Announcement
- Properties of the autocorrelation
- The power spectrum
- LTI systems and the power spectrum
- "White noise"
- Introduction to z-transform
- Examples
  - Finite-length
  - Left-sided, Right-sided, Two-sided

Announcement

- Quiz I
- Thursday, Feb. 16, during class hour.
- It will be open book, one page (double sided) formula sheet,
- Closed HWK solutions.
- Coverage: Chapters 2 and 3.

Properties of the Autocorrelation

- Definition:

$$\phi_{xx}[m] = E\left\{x[n+m]x^*[n]\right\}$$

- Average power:

$$\phi_{xx}[0] = E\left\{|x[n]|^2\right\} = \text{mean} - \text{square}$$

- Symmetry:

$$\phi_{xx}[-m] = \phi_{xx}^*[m] \qquad \phi_{xx}[-m] = \phi_{xx}[m] \quad \text{if } x \text{ is real}$$

- Shape:

$$|\phi_{xx}[m]| \le \phi_{xx}[0] \qquad \lim_{m\to\infty} \phi_{xx}[m] = |m_x|^2$$

Power Density Spectrum

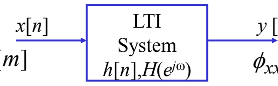
- The DTFT of a random signal does not exist theoretically. However, the autocorrelation function of the signal does exist, and the power density spectrum of a random signal is the DTFT of the autocorrelation function.

$$\Phi(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m]e^{-j\omega m}$$

$$\phi_{xx}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(e^{j\omega})e^{j\omega m}d\omega$$

$$E\left\{|x|^2\right\} = \phi_{xx}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(e^{j\omega})d\omega$$

Linear System with a Random Input



$\phi_{xx}[m]$ $\quad\quad$ $\phi_{xx}[m] * c_{hh}[m]$

$\Phi_{xx}(e^{j\omega})$ $\quad\quad$ $\Phi_{xx}(e^{j\omega})C_{hh}(e^{j\omega})$

$$\phi_{yy}[m] = \phi_{xx}[m] * c_{hh}[m]$$

$$c_{hh}[m] = \sum_{k=-\infty}^{\infty} h[m+k]h^*[k] = h[-m] * h^*[m]$$

$$\Phi_{yy}(e^{j\omega}) = \Phi_{xx}(e^{j\omega})C_{hh}(e^{j\omega})$$

$$C_{hh}(e^{j\omega}) = H(e^{-j\omega})H^*(e^{-j\omega}) = \left|H(e^{-j\omega})\right|^2$$

Properties of Power Density

- Real:

$$\Phi^*(e^{j\omega}) = \Phi(e^{j\omega})$$

- Symmetry:

$$\Phi(e^{j\omega}) = \Phi(e^{-j\omega}) \quad \text{if } x[n] \text{ is real}$$

- Positivity:

$$\Phi(e^{j\omega}) \ge 0$$

- Magnitude-squared has same properties:

$$C^*_{hh}(e^{j\omega}) = \left|H(e^{-j\omega})\right|^2 = C_{hh}(e^{j\omega}) \quad \Rightarrow \quad \text{real}$$

$$C_{hh}(e^{-j\omega}) = \left|H(e^{j\omega})\right|^2 = C_{hh}(e^{j\omega}) \quad \text{if } x[n] \text{ is real}$$

White Noise

- Consider a zero mean signal whose autocorrelation function is

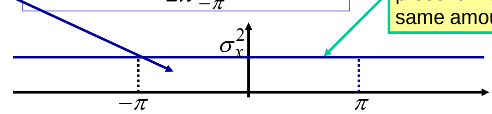
$$\phi_{xx}[m] = \sigma_x^2\delta[m]$$

- The power spectrum of this signal is

$$\Phi_{xx}[m] = \sigma_x^2 \quad |\omega| \le \pi$$

$$\text{ave. power} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sigma_x^2 d\omega = \sigma_x^2$$

All frequencies present in the same amount.



Linear System with a White Noise Input

$x[n]$ → LTI System $h[n], H(e^{j\omega})$ → $y[n]$

$\phi_{xx}[m] = \sigma_x^2 \delta[m]$ $\qquad$ $\phi_{yy}[m] = \sigma_x^2 c_{hh}[m]$

$\Phi_{xx}(e^{j\omega}) = \sigma_x^2$ $\qquad$ $\Phi_{yy}(e^{j\omega}) = \sigma_x^2 C_{hh}(e^{j\omega})$

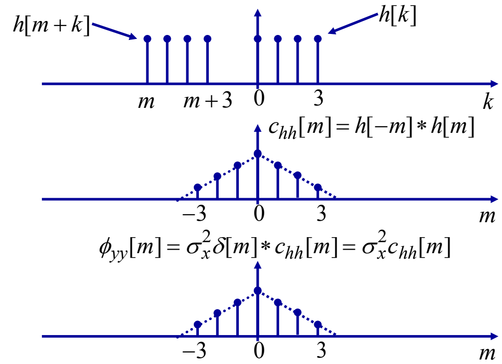
$$\phi_{yy}[m] = \sigma_x^2 \delta[m] * c_{hh}[m] = \sigma_x^2 c_{hh}[m]$$

$$c_{hh}[m] = \sum_{k=-\infty}^{\infty} h[m+k]h^*[k] = h[-m] * h^*[m]$$

$$\Phi_{yy}(e^{j\omega}) = \Phi_{xx}(e^{j\omega}) C_{hh}(e^{j\omega}) = \sigma_x^2 C_{hh}(e^{j\omega})$$

$$C_{hh}(e^{j\omega}) = H(e^{-j\omega}) H^*(e^{-j\omega}) = \left|H(e^{-j\omega})\right|^2$$

White Noise into Moving Average



White Noise into Moving Average

- Frequency response of filter:

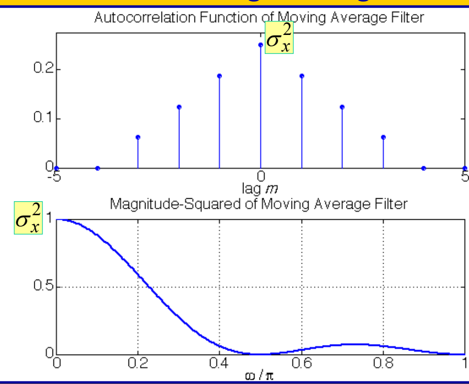
$$H(e^{j\omega}) = \frac{1}{M+1} \sum_{n=0}^{M} e^{-j\omega n} = \frac{\sin[(M+1)\omega/2]}{(M+1)\sin(\omega/2)} e^{-j\omega M/2}$$

- Power spectrum of output when the input is white noise:

$$\Phi_{yy}(e^{j\omega}) = \sigma_x^2 \left|H(e^{j\omega})\right|^2$$

$$= \sigma_x^2 \left(\frac{\sin[(M+1)\omega/2]}{(M+1)\sin(\omega/2)} \right)^2$$

4-Point Moving Average Filter



The *z*-Transform

- The *z*-Transform of a sequence is defined as

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

- Since this is generally an infinite sum, we need to be concerned about "convergence"; i.e., is the sum finite? In general convergence will depend upon *z*; e.g., $0\le r_R<|z|<r_L<\infty$.
- The inverse *z*-transform is given by the contour integral $$x[n]=\frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

Region of Convergence

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

$$|X(z)|=\left|\sum_{n=-\infty}^{\infty}x[n]z^{-n}\right|\le\sum_{n=-\infty}^{\infty}|x[n]||z|^{-n}$$

- The region of convergence is the set of values of *z* $0\le r_R<|z|<r_L<\infty$ such that

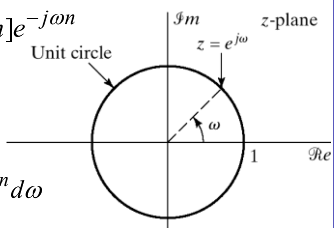
$$\sum_{n=-\infty}^{\infty}|x[n]||z|^{-n}<\infty$$

$|z|^{-n}$ can "tame" a growing sequence

Relation to DTFT

- The DTFT is equal to the z-transform evaluated on the unit circle:

$$X(z)|_{z=e^{j\omega}}=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$
$$=X(e^{j\omega})$$
$$=\text{DTFT}$$



$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$$

$$\sum_{n=-\infty}^{\infty}|x[n]|<\infty\Rightarrow\text{ ROC contains }|z|=1$$

Remember These Formulas

- A tricky manipulation:

$$z(a_0+a_1z^{-1}+a_0z^{-2})=a_0z+a_1+a_0z^{-1}$$
$$=a_1+a_0(z+z^{-1})$$

- Summing a finite exponential sequence:

$$\sum_{n=N_1}^{N_2}\alpha^n=\frac{\alpha^{N_1}-\alpha^{N_2+1}}{1-\alpha}\qquad N_2\ge N_1$$

- Summing an infinite exponential sequence:

$$\sum_{n=0}^{\infty}\alpha^n=\frac{1}{1-\alpha}\qquad\text{if }|\alpha|<1$$

Examples

- Impulse sequence: $x[n] = \delta[n - n_0]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] z^{-n} = z^{-n_0}$$

- Pulse sequence: $x[n] = \begin{cases} 1 & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$

$$X(z) = \sum_{n=0}^{M-1} z^{-n} = \frac{1 - z^{-M}}{1 - z^{-1}} \qquad z \neq 0$$

Right-Sided Exponential Signal

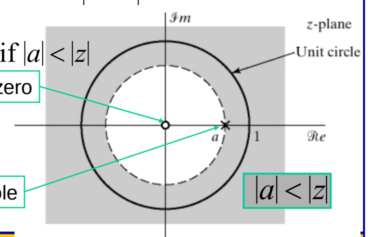
- Right-sided exponential sequence: $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - (az^{-1})} \quad \text{if } |az^{-1}| < 1$$

$$X(z) = \frac{1}{1 - az^{-1}} \quad \text{if } |a| < |z|$$

$$= \frac{(z - 0)}{(z - a)}$$



Left-Side Exponential Signal

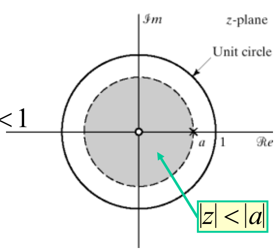
- Left-sided exponential sequence: $x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$= 1 - \frac{1}{1 - a^{-1}z} \quad \text{if } |a^{-1}z| < 1$$

$$= \frac{1 - a^{-1}z - 1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}}$$



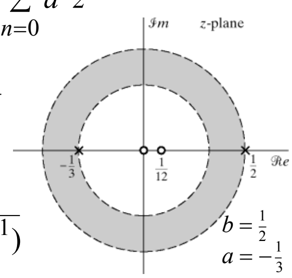
Two-Sided Exponential Signal

$$x[n] = -b^n u[-n-1] + a^n u[n]$$

$$X(z) = -\sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \underbrace{\frac{1}{1 - bz^{-1}}}_{\text{if } |z|<|b|} + \underbrace{\frac{1}{1 - az^{-1}}}_{\text{if } |z|>|a|}$$

$$= \frac{2 - (a+b)z^{-1}}{(1 - az^{-1})(1 - bz^{-1})} \quad \text{if } |a|<|z|<|b|$$



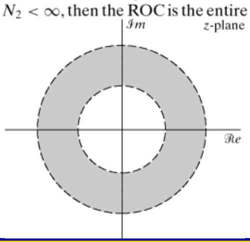
Properties of the z-Transform - I

PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.



Properties of the z-Transform - II

PROPERTY 5: If $x[n]$ is a *right-sided sequence,* i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence,* i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

Transform Pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$

Transform Pairs

Sequence	Transform	ROC
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$	$\|z\| > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
10	3.4.8	Initial-value theorem:		
		$x[n] = 0, \quad n < 0 \qquad \lim_{z\to\infty} X(z) = x[0]$		