ECE4270
Fundamentals of DSP

Lecture 9

Z-Transform

School of ECE
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Overview of Lecture

- Examples
  - Two-sided
- Properties of the Region of Convergence
- Transform Pairs
- Uniqueness
- Partial Fraction Expansion
- Partial Fraction Expansion (PFE) Example
- *Repeated Roots in PFE*
- z-transform Theorems
- System Function of a DE

The *z*-Transform (previous lecture)

- The *z*-Transform of a sequence is defined as

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}}$$

- The region of convergence is the set of values of z $0 \le r_R < |z| < r_L < \infty$ such that

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty}$$

$|z|^{-n}$ can "tame" a growing sequence

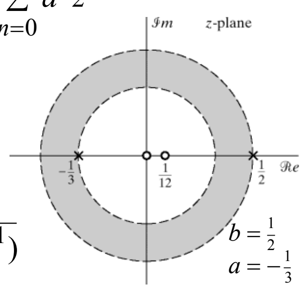
Two-Sided Exponential Signal

$$x[n] = -b^n u[-n-1] + a^n u[n]$$

$$X(z) = -\sum_{n=-\infty}^{-1} b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \underset{\text{if } |z|<|b|}{\frac{1}{1-bz^{-1}}} + \underset{\text{if } |z|>|a|}{\frac{1}{1-az^{-1}}}$$

$$= \underset{\text{if } |a|<|z|<|b|}{\frac{2-(a+b)z^{-1}}{(1-az^{-1})(1-bz^{-1})}}$$



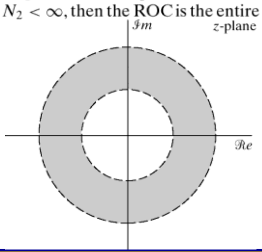
Properties of the z-Transform - I

PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.



Properties of the z-Transform - II

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

Transform Pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$

Transform Pairs

Sequence	Transform	ROC
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\| R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
10	3.4.8	Initial-value theorem:		
		$x[n] = 0, \quad n < 0$ $\quad \lim_{z\to\infty} X(z) = x[0]$		

Partial Fraction Expansion - I

- Consider a general rational z-transform

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

- We can find a partial fraction expansion in the form

$$X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{only if } M \ge N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

Partial Fraction Expansion - II

$$X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{long division gets us this}} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

$$(1 - d_m z^{-1}) X(z) = (1 - d_m z^{-1}) \left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right] + \sum_{k=1}^{N} \frac{(1 - d_m z^{-1}) A_k}{1 - d_k z^{-1}}$$

Partial Fraction Expansion - III

$$(1 - d_m z^{-1}) X(z)\Big|_{z=d_m} = \cancelto{0}{(1 - d_m z^{-1})} \left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]_{z=d_m} + \sum_{k=1}^{N} \frac{\cancelto{0}{(1 - d_m z^{-1})} A_k}{1 - d_k z^{-1}}\Bigg|_{z=d_m} = A_m$$

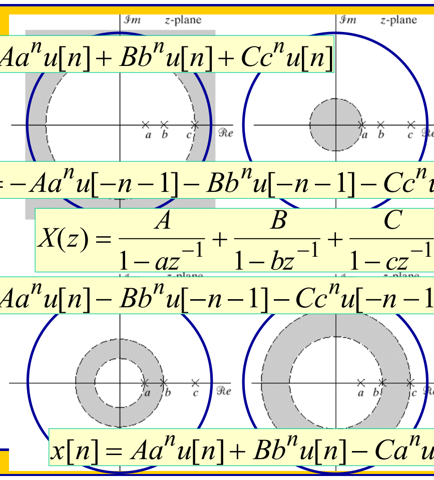
$$\boxed{X(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{if } M \ge N} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}}$$

$$\boxed{A_k = (1 - d_k z^{-1}) X(z)\Big|_{z=d_k}}$$

Partial Fraction Expansion - IV

$$X(z)=\underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{if } M\geq N}+\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$$

$$x[n]=\underbrace{\left[\sum_{r=0}^{(M-N)} B_r \delta[n-r]\right]}_{\text{if } M\geq N}+\underbrace{\sum_k A_k d_k^n u[n]}_{\text{when } |d_k|<r_R}-\underbrace{\sum_k A_k d_k^n u[-n-1]}_{\text{when } |d_k|>r_L}$$



$$x[n]=Aa^n u[n]+Bb^n u[n]+Cc^n u[n]$$

$$x[n]=-Aa^n u[-n-1]-Bb^n u[-n-1]-Cc^n u[-n-1]$$

$$X(z)=\frac{A}{1-az^{-1}}+\frac{B}{1-bz^{-1}}+\frac{C}{1-cz^{-1}}$$

$$x[n]=Aa^n u[n]-Bb^n u[-n-1]-Cc^n u[-n-1]$$

$$x[n]=Aa^n u[n]+Bb^n u[n]-Ca^n u[-n-1]$$

Long Division

$$X(z)=\frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+2z^{-2}}=\frac{(1+z^{-1})^3}{(1-z^{-1})(1-2z^{-1})}$$

$$\begin{array}{r|l}
 & 0.5z^{-1}+2.25 \\ \hline
2z^{-2}-3z^{-1}+1 & z^{-3}+3.0z^{-2}+3.0z^{-1}+1.0 \\
 & z^{-3}-1.5z^{-2}+0.5z^{-1} \\ \hline
 & \quad 4.5z^{-2}+2.50z^{-1} \\
 & \quad 4.5z^{-2}-6.75z^{-1}+2.25 \\ \hline
 & \qquad 9.25z^{-1}-1.25
\end{array}$$

$$X(z)=\frac{(1+z^{-1})^3}{(1-z^{-1})(1-2z^{-1})}=2.25+0.5z^{-1}-\frac{1.25-9.25z^{-1}}{(1-z^{-1})(1-2z^{-1})}$$

Partial Fraction Expansion

$$X(z)=\frac{(1+z^{-1})^3}{(1-z^{-1})(1-2z^{-1})}$$

$$=2.25+0.5z^{-1}+\frac{A_1}{(1-z^{-1})}+\frac{A_2}{(1-2z^{-1})}$$

$$A_1=X(z)(1-z^{-1})\Big|_{z=1}=\frac{(1+z^{-1})^3}{(1-2z^{-1})}\Bigg|_{z=1}=\frac{8}{-1}=-8$$

$$A_2=X(z)(1-2z^{-1})\Big|_{z=2}=\frac{(1+z^{-1})^3}{(1-z^{-1})}\Bigg|_{z=2}=\frac{(3/2)^3}{1/2}=6.75$$

$$X(z)=2.25+0.5z^{-1}+\frac{-8}{(1-z^{-1})}+\frac{6.75}{(1-2z^{-1})}$$

Writing Down $x[n]$

$$X(z) = 2.25 + 0.5z^{-1} + \frac{-8}{(1-z^{-1})} + \frac{6.75}{(1-2z^{-1})}$$

- If ROC is

$$2 < |z|$$

$$x[n] = 2.25\delta[n] + 0.5\delta[n-1] - 8u[n] + 6.75(2)^n u[n]$$

- If ROC is

$$1 < |z| < 2$$

$$x[n] = 2.25\delta[n] + 0.5\delta[n-1] - 8u[n] - 6.75(2)^n u[-n-1]$$

- If ROC is

$$|z| < 1$$

$$x[n] = 2.25\delta[n] + 0.5\delta[n-1] + 8u[-n-1] - 6.75(2)^n u[-n-1]$$

Partial Fraction Expansion in MATLAB

$$X(z) = \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+2z^{-2}} = 2.25 + 0.5z^{-1} + \frac{-8}{1-z^{-1}} + \frac{6.75}{1-2z^{-1}}$$

```
» [r,p,k]=residuez([1,3,3,1],[1,-3,2])
r =
   6.75000000000000
  -8.00000000000000
p =
     2
     1
k =
   2.25000000000000   0.50000000000000
```


Repeated Roots in PFE

Suppose ***H(z)*** has a pole of order ***s*** at ***z=d$_i$*** and all the other poles are first order:

$$H(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{only if } M \ge N} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1-d_i z^{-1})^m}$$

- The coefficients A_k and B_r are found as before. C_m is found by:

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} \left[(1-d_i w)^s H(w^{-1})\right] \right\}_{w=d_i^{-1}}$$

(s-m)th order derivative

for $m = 1, \ldots, s$

z-Transform Theorems

- The delay or shift property:

$$x[n-n_0] \quad \Leftrightarrow \quad z^{-n_0} X(z)$$

- The convolution property:

$$y[n] = x[n] * h[n] \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

- Example:

$$y[n] = x[n] * \delta[n-n_0] \quad \Leftrightarrow \quad Y(z) = z^{-n_0} X(z)$$

An IIR System

- Difference equation:

$$y[n] = ay[n-1] + x[n] \Leftrightarrow Y(z) = az^{-1}Y(z) + X(z)$$

- System function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z|>|a|$$

ROC if causal

- Frequency response:

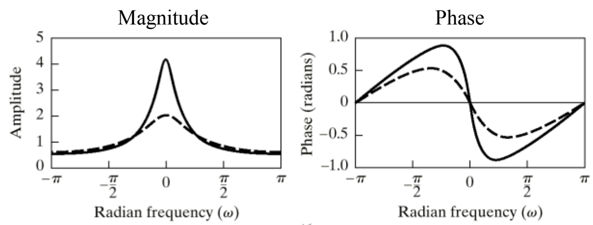
$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{1}{1-ae^{-j\omega}}$$

Assumes |a|<1

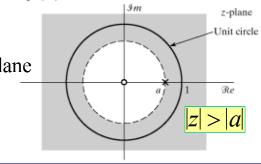
- Impulse response:

$$h[n] = a^n u[n]$$

IIR Frequency Response



z-plane



System Function of a DE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\sum_{k=0}^{M} b_k z^{-k}\right)}{\left(\sum_{k=0}^{N} a_k z^{-k}\right)}$$

Frequency Response of a DE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right)}{\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right)}$$

ROC must Contain the Unit circle

LTI Systems

	LTI System	
$x[n]$	$\longrightarrow$	$y[n]$
$\delta[n]$		$h[n]$
$e^{j\omega n}$		$H(e^{j\omega})e^{j\omega n}$
$X(e^{j\omega})$		$X(e^{j\omega})H(e^{j\omega})$
$X(z)$		$X(z)H(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\sum_{k=0}^{M} b_k z^{-k}\right)}{\left(\sum_{k=0}^{N} a_k z^{-k}\right)}$$

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$