ECE4270
Fundamentals of DSP

Lecture 10

z-Transform & Sampling and Reconstruction from Samples

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Overview of Lecture

- *Repeated Roots in PFE*
- z-transform Theorems
- System Function of a DE
- Causality, stability and ***z***-transform
- Chapter 4--Sampling
  - Bandlimited signals
  - Sampling a sine wave
  - Ideal Continuous-to-Discrete conversion (sampling)
  - Derivation of frequency-domain formula
  - Interpretation of frequency-domain formula
    - Oversampling
    - Aliasing distortion

Repeated Roots in PFE

Suppose ***H(z)*** has a pole of order ***s*** at ***z=d_i*** and all the other poles are first order:

$$H(z) = \underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{only if } M \geq N} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1-d_i z^{-1})^m}$$

- The coefficients A_k and B_r are found as before. C_m is found by:

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \underbrace{\frac{d^{s-m}}{dw^{s-m}}}_{(s-m)^{th}\text{ order derivative}} [(1-d_i w)^s H(w^{-1})] \right\}_{w=d_i^{-1}}$$

for $m = 1, \ldots, s$

z-Transform Theorems

- The delay or shift property:

$$\boxed{x[n-n_0] \quad \Leftrightarrow \quad z^{-n_0} X(z)}$$

- The convolution property:

$$\boxed{y[n] = x[n] * h[n] \quad \Leftrightarrow \quad Y(z) = H(z)X(z)}$$

- Example:

$$y[n] = x[n] * \delta[n-n_0] \quad \Leftrightarrow \quad Y(z) = z^{-n_0} X(z)$$

An IIR System

- Difference equation:

$$y[n] = ay[n-1] + x[n] \Leftrightarrow Y(z) = az^{-1}Y(z) + X(z)$$

- System function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z| > |a|$$

ROC if causal

- Frequency response:

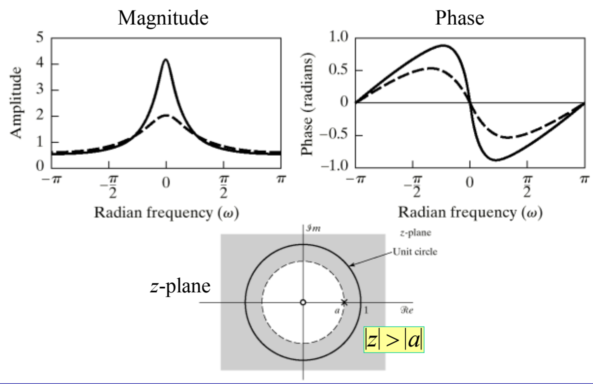
$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{1}{1-ae^{-j\omega}}$$

Assumes |a|<1

- Impulse response:

$$h[n] = a^n u[n]$$

IIR Frequency Response



System Function of a DE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\sum_{k=0}^{M} b_k z^{-k}\right)}{\left(\sum_{k=0}^{N} a_k z^{-k}\right)}$$

Frequency Response of a DE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{\left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right)}{\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right)}$$

ROC must Contain the Unit circle

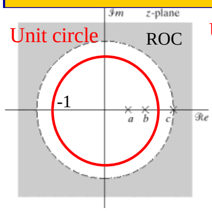
LTI Systems

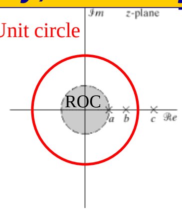
	LTI System	
$x[n]$	→	$y[n]$
$\delta[n]$	→	$h[n]$
$e^{j\omega n}$		$H(e^{j\omega})e^{j\omega n}$
$X(e^{j\omega})$		$X(e^{j\omega})H(e^{j\omega})$
$X(z)$		$X(z)H(z)$

$$\boxed{H(z)=\frac{Y(z)}{X(z)}=\frac{\left(\sum_{k=0}^{M} b_k z^{-k}\right)}{\left(\sum_{k=0}^{N} a_k z^{-k}\right)}}$$

$$\boxed{H(e^{j\omega})=H(z)\big|_{z=e^{j\omega}}}$$

Stability, Causality, ROC





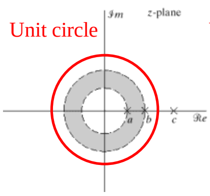
- May be Causal
- Not stable

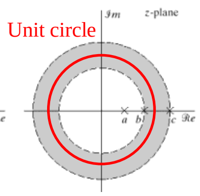
- Not Causal
- Not stable

Stability, Causality, ROC

- Not Causal
- Not stable

- Not Causal
- Stable





ROC of Z-Transform and Causality

ROC of Z-Transform and Causality

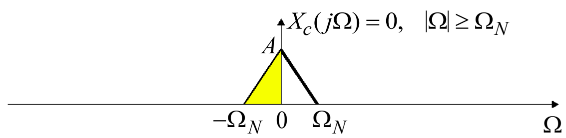
ROC of Z-Transform and Causality

Digital Processing of Analog Signals

- **A-to-D conversion:** sampling and quantization
- **Numerical algorithm:** convolution, difference equations, DFT, LPC
  - Implemented on DSP chips, computers or ASICs with finite-precision arithmetic
- **D-to-A conversion:** quantization and filtering

Bandlimited Signals

- A bandlimited continuous-time (analog) signal has a Fourier transform that is zero for all frequencies above some highest frequency (called the *Nyquist* frequency Ω_N).



- Simple example is a sinusoidal signal

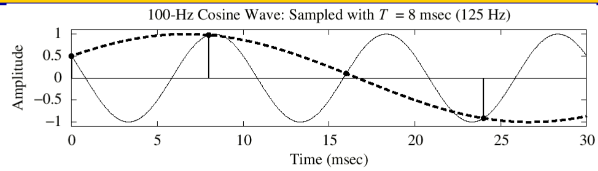
$$x_c(t) = A\cos(\Omega_0 t + \phi) \qquad \Omega_N = \Omega_0 + \varepsilon$$

Sampling a Sinusoidal Signal

$x_c(t) \rightarrow$ **C-to-D Converter** $\rightarrow x[n] = x_c(nT)$

$$x_1(t) = A\cos(\Omega_0 t + \phi)$$
$$x_1[n] = \underline{A\cos(\Omega_0 nT + \phi)} = A\cos(\omega_0 n + \phi) \qquad \omega_0 = \Omega_0 T$$
$$x_2(t) = A\cos[(\Omega_0 + 2\pi/T)t + \phi]$$
$$\begin{aligned} x_2[n] &= A\cos[(\Omega_0 + 2\pi/T)nT + \phi)] \\ &= A\cos(\Omega_0 nT + 2\pi n + \phi) = \underline{A\cos(\Omega_0 nT + \phi)} \end{aligned}$$
$$x_3(t) = A\cos[(2\pi/T - \Omega_0)t - \phi]$$
$$\begin{aligned} x_3[n] &= A\cos[(2\pi/T - \Omega_0)nT - \phi)] \\ &= A\cos(2\pi n - \Omega_0 nT - \phi) = \underline{A\cos(\Omega_0 nT + \phi)} \end{aligned}$$

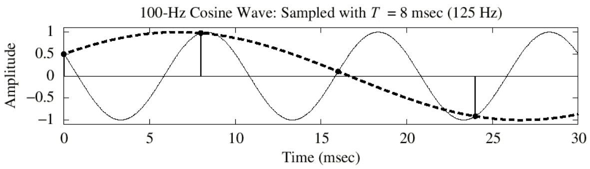
Aliasing and Reconstruction



- Each of the sinusoidal components of a signal has an infinite number of aliases.
- The ideal D-to-C converter reconstructs the lowest frequency alias of each sinusoidal component. This is why we need to constrain our sampling rate so that

$$\frac{2\pi}{T} \geq 2\Omega_N$$

The Sampling Theorem



A bandlimited signal with Nyquist (highest) frequency Ω_N can be reconstructed exactly from samples taken with sampling frequency $\frac{2\pi}{T} \geq 2\Omega_N$

Sampling (C-to-D Conversion)

$x_c(t)$, $X_c(j\Omega)$ $\rightarrow$ **C-to-D Converter** $\rightarrow$ $x[n] = x_c(nT)$, $X(e^{j\omega}), X(e^{j\Omega T})$

- Discrete-time Fourier transform:

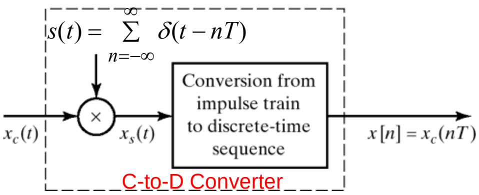
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Frequency-domain relation:

$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega T n} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

- Sampling frequency: $\Omega_s = 2\pi/T$
- Normalized frequency: $\omega = \Omega T$

Derivation of Basic FT Formula - I



$$x_s(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT)$$

$$\boxed{x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)}$$

Derivation of Basic FT Formula - II

$$\boxed{x_s(t) = x_c(t) \cdot s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)}$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \Leftrightarrow S(j\Omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta(\Omega - k\frac{2\pi}{T})$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta(\Omega - k\frac{2\pi}{T})$$

$$\boxed{X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(\Omega - k\frac{2\pi}{T})}$$

Derivation of Basic FT Formula - III

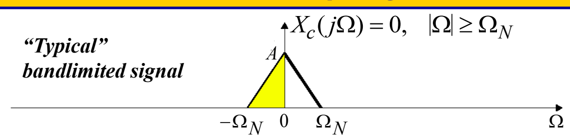
$$\boxed{x_s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)}$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega nT}$$

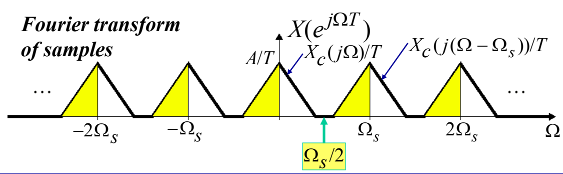
$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\Omega T)n} = X(e^{j\Omega T})$$

$$\boxed{X(e^{j\Omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(\Omega - k\frac{2\pi}{T})}$$

Oversampling

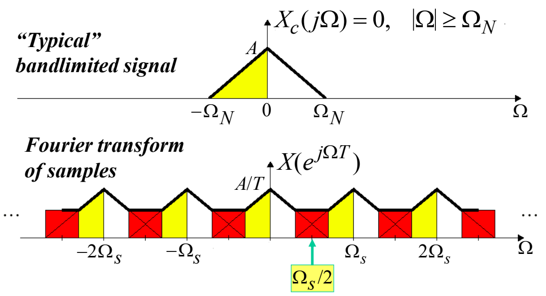


$$\boxed{X(e^{j\Omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \qquad \Omega_s = 2\pi/T}$$



Undersampling (Aliasing Distortion)

- If $\Omega_s < 2\Omega_N$, the copies of $X_c(j\Omega)$ overlap, and we have aliasing distortion.



Sampling Theorem

$x_c(t)$, $X_c(j\Omega)$ → **C-to-D Converter** → $x[n] = x_c(nT)$, $X(e^{j\omega}), X(e^{j\Omega T})$

- Frequency-domain representation

$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega T n} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

- Sampling theorem justification:

If $X_c(j\Omega) = 0, \quad |\Omega| \geq \Omega_N$ and $\Omega_s/2 = \pi/T \geq \Omega_N$,

- Therefore we should be able to recover $x_c(t)$!

then $X(e^{j\Omega T}) = \frac{1}{T}X_c(j\Omega), \quad |\Omega| \leq \pi/T = \Omega_s/2$