ECE4270 Fundamentals of DSP

Lecture 11

Sampling and Reconstruction from Samples

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Overview of Lecture

- Chapter 4--Sampling
  - Bandlimited signals
  - Ideal Continuous-to-Discrete conversion (sampling)
  - Derivation of frequency-domain formula
  - Interpretation of frequency-domain formula
    - Oversampling
    - Aliasing distortion
- Band Limited Reconstruction

Sampling (C-to-D Conversion)

$x_c(t)$, $X_c(j\Omega)$ → **C-to-D Converter** → $x[n]=x_c(nT)$, $X(e^{j\omega}), X(e^{j\Omega T})$

- Discrete-time Fourier transform:

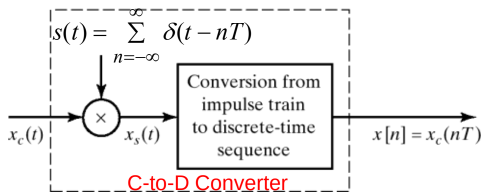
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Frequency-domain relation:

$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega T n} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

- Sampling frequency: $\Omega_s = 2\pi/T$
- Normalized frequency: $\omega = \Omega T$

Derivation of Basic FT Formula - I



$$x_s(t) = x_c(t)\cdot \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t-nT)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)$$

Derivation of Basic FT Formula - II

$$x_s(t) = x_c(t)\cdot s(t) = x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$s(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT) \Leftrightarrow S(j\Omega) = \sum_{k=-\infty}^{\infty}\frac{2\pi}{T}\delta(\Omega - k\frac{2\pi}{T})$$

$$X_s(j\Omega) = \frac{1}{2\pi}X_c(j\Omega) * S(j\Omega)$$

$$X_s(j\Omega) = \frac{1}{2\pi}X_c(j\Omega) * \sum_{k=-\infty}^{\infty}\frac{2\pi}{T}\delta(\Omega - k\frac{2\pi}{T})$$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(\Omega - k\frac{2\pi}{T})$$

Derivation of Basic FT Formula - III

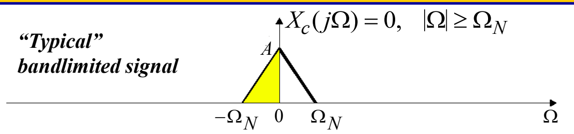
$$x_s(t) = x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty}x[n]\delta(t-nT)$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega nT}$$

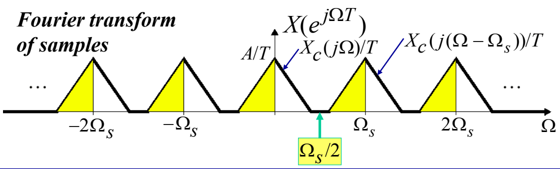
$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty}x[n]e^{-j(\Omega T)n} = X(e^{j\Omega T})$$

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(\Omega - k\frac{2\pi}{T})$$

Oversampling

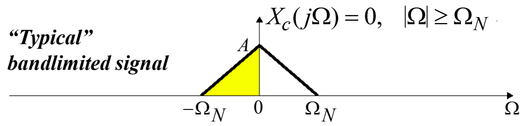


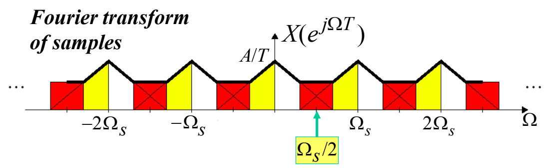
$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j(\Omega - k\Omega_s)) \qquad \Omega_s = 2\pi/T$$



Undersampling (Aliasing Distortion)

- If $\Omega_s < 2\Omega_N$, the copies of $X_c(j\Omega)$ overlap, and we have aliasing distortion.





Sampling Theorem

$x_c(t)$, $X_c(j\Omega)$ → **C-to-D Converter** → $x[n] = x_c(nT)$, $X(e^{j\omega}), X(e^{j\Omega T})$

- Frequency-domain representation

$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega T n} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

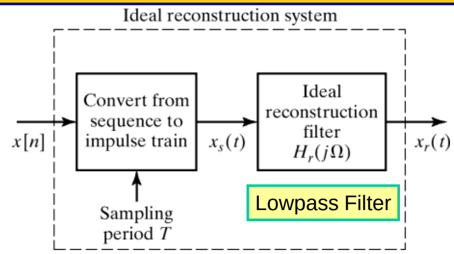
- Sampling theorem justification:

If $X_c(j\Omega) = 0, \quad |\Omega| \geq \Omega_N \quad$ and $\quad \Omega_s/2 = \pi/T \geq \Omega_N$,

- Therefore we should be able to recover $x_c(t)$!

then $X(e^{j\Omega T}) = \frac{1}{T}X_c(j\Omega), \quad |\Omega| \leq \pi/T = \Omega_s/2$

Reconstruction from Samples (I)



$$x_r(t) = x_s(t) * h_r(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT) * h_r(t)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT)$$

Reconstruction from Samples (II)

$x[n] = x_c(nT)$, $X(e^{j\omega}), X(e^{j\Omega T})$ → **D-to-C Converter** → $x_r(t)$, $X_r(j\Omega)$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT)$$

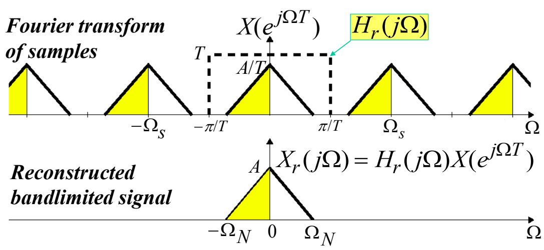
- Frequency-domain representation

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]\left(e^{-j\Omega T n}H_r(j\Omega)\right)$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega T n}\right)H_r(j\Omega) = X(e^{j\Omega T})H_r(j\Omega)$$

$$= \left(\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k/T))\right)H_r(j\Omega)$$

Bandlimited Reconstruction (I)



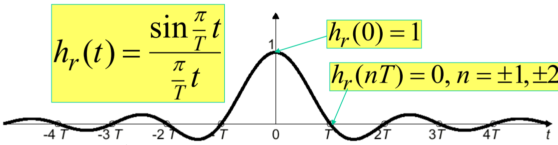
- If $\Omega_s \geq 2\Omega_N$, the copies of $X_c(j\Omega)$ do not overlap, so $X_r(j\Omega) = H_r(j\Omega)\,X(e^{j\Omega T}) = X_c(j\Omega)$, and we get perfect reconstruction.

Bandlimited Reconstruction (II): Reconstruction Filter

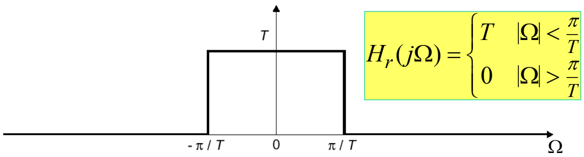
$$h_r(t) = \frac{\sin \frac{\pi}{T} t}{\frac{\pi}{T} t}$$

$h_r(0) = 1$

$h_r(nT) = 0, \; n = \pm 1, \pm 2, \ldots$



$$H_r(j\Omega) = \begin{cases} T & |\Omega| < \frac{\pi}{T} \\ 0 & |\Omega| > \frac{\pi}{T} \end{cases}$$



Bandlimited Reconstruction (III): (Time-Domain)

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) h_r(t - nT) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$