ECE4270 Fundamentals of DSP

Lecture 11

Sampling/Reconstruction & DT Filtering of CT Signals

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Overview of Lecture

- Sampling (overview of last lecture)
- Band Limited Reconstruction (overview of last lecture)
- Zero-order hold reconstruction (D-to-A converter)
- Linear interpolation
- Comparison of Interpolation Filters
- DT filtering of CT signals
  - Derivation of general formula
  - Illustration
  - Examples

Sampling (C-to-D Conversion)

$x_c(t)$, $X_c(j\Omega)$ → **C-to-D Converter** → $x[n] = x_c(nT)$, $X(e^{j\omega}), X(e^{j\Omega T})$

- Discrete-time Fourier transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

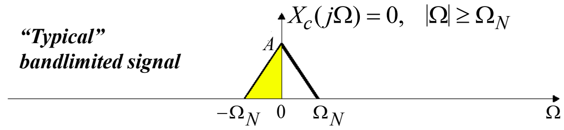
- Frequency-domain relation:

$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega T n} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

- Sampling frequency: $\Omega_s = 2\pi/T$
- Normalized frequency: $\omega = \Omega T$

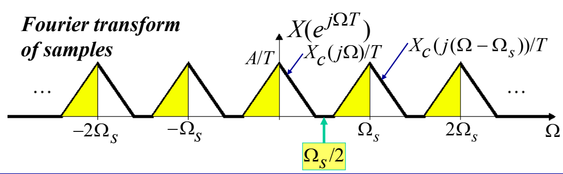
Oversampling

"Typical" bandlimited signal



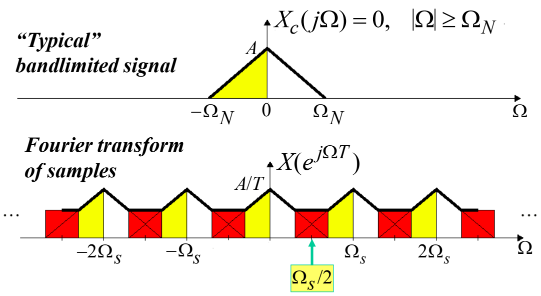
$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \qquad \Omega_s = 2\pi/T$$

Fourier transform of samples

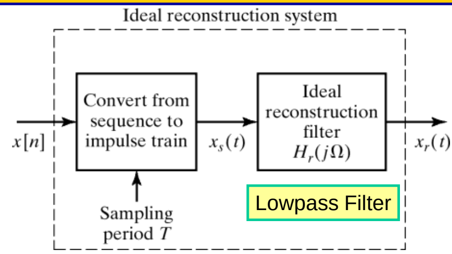


Undersampling (Aliasing Distortion)

- If $\Omega_s < 2\Omega_N$, the copies of $X_c(j\Omega)$ overlap, and we have aliasing distortion.



Reconstruction from Samples (I)



$$x_r(t) = x_s(t) * h_r(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT) * h_r(t)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t-nT)$$

Reconstruction from Samples (II)

$x[n] = x_c(nT)$, $X(e^{j\omega}), X(e^{j\Omega T})$ → **D-to-C Converter** → $x_r(t)$, $X_r(j\Omega)$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t-nT)$$

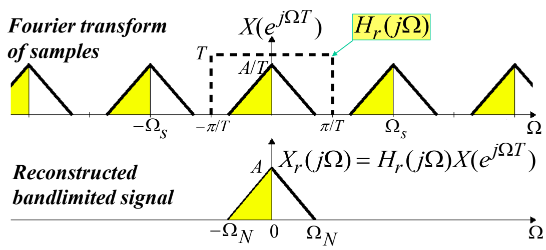
- Frequency-domain representation

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]\left(e^{-j\Omega Tn} H_r(j\Omega)\right)$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega Tn}\right) H_r(j\Omega) = X(e^{j\Omega T})H_r(j\Omega)$$

$$= \left(\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k/T))\right) H_r(j\Omega)$$

Bandlimited Reconstruction (I)



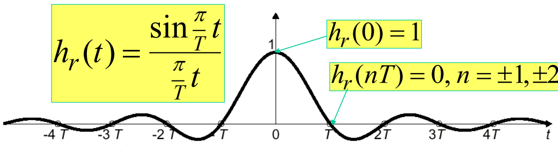
- If $\Omega_s \ge 2\Omega_N$, the copies of $X_c(j\Omega)$ do not overlap, so $X_r(j\Omega) = H_r(j\Omega)\, X(e^{j\Omega T}) = X_c(j\Omega)$, and we get perfect reconstruction.

Bandlimited Reconstruction (II): Reconstruction Filter

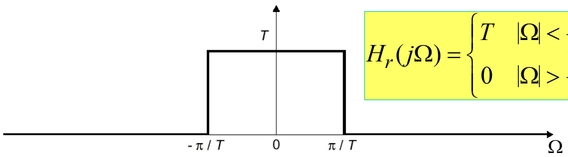
$$h_r(t) = \frac{\sin \frac{\pi}{T} t}{\frac{\pi}{T} t}$$

$h_r(0) = 1$

$h_r(nT) = 0, n = \pm 1, \pm 2, \ldots$

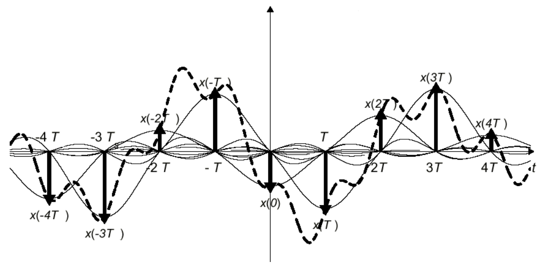


$$H_r(j\Omega) = \begin{cases} T & |\Omega| < \frac{\pi}{T} \\ 0 & |\Omega| > \frac{\pi}{T} \end{cases}$$



Bandlimited Reconstruction (III): (Time-Domain)

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) h_r(t - nT) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$



Signal Reconstruction

$x[n] = x_c(nT)$ → **D-to-C Converter** → $x_r(t)$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT)$$

- Types of interpolation pulses:
  - Square pulse -- holds sample value (D/A in practice)
  - Triangular pulse -- linear interpolation
  - Sinc pulse -- ideal bandlimited interpolation

$$h_r(t) = \frac{\sin(\pi t / T)}{\pi t / T}$$

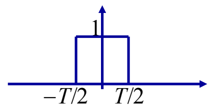
D-to-A Conversion

$\hat{x}[n]$ → **D-to-A Converter** → $x_{DA}(t)$

aka zero-order hold

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n] h_0(t - nT), \quad \text{where} \quad h_0(t) = \begin{cases} 1, & -T/2 \le t \le T/2 \\ 0, & \text{otherwise} \end{cases}$$

$$X_{DA}(j\Omega) = \hat{X}(e^{j\Omega T}) \underbrace{\frac{2\sin(\Omega T/2)}{\Omega}}_{H_0(j\Omega)}$$



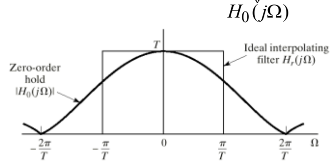
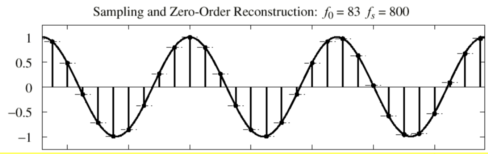
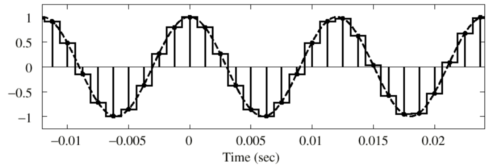


Illustration of D-to-A Conversion

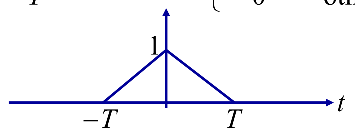


$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n]h_0(t-nT), \quad \text{where} \quad h_0(\text{t}) = \begin{cases} 1, & -T/2 \le t \le T/2 \\ 0, & \text{otherwise} \end{cases}$$



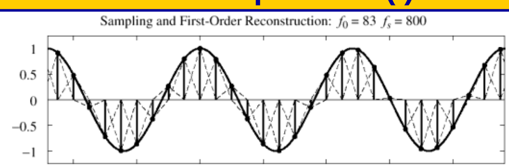
Linear Interpolation

$$h_1(t) = \frac{1}{T}h_0(t) * h_0(t) = \begin{cases} 1-|t/T| & |t| < T/2 \\ 0 & \text{otherwise} \end{cases}$$

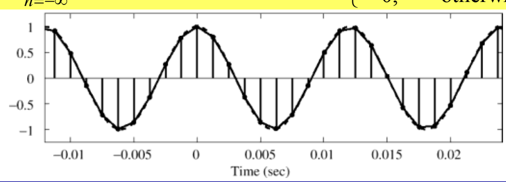


$$H_1(j\Omega) = \frac{1}{T}\left(\frac{\sin(\Omega T/2)}{(\Omega/2)}\right)\left(\frac{\sin(\Omega T/2)}{(\Omega/2)}\right) = T\left(\frac{\sin(\Omega T/2)}{(\Omega T/2)}\right)^2$$

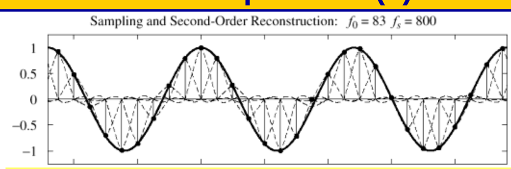
Linear Interpolation (I)



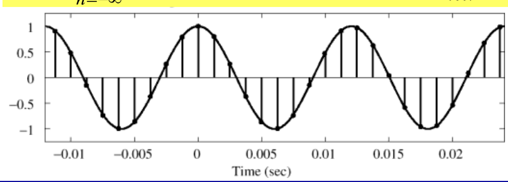
$$x_r(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n]h_1(t-nT), \quad \text{where} \quad h_1(\text{t}) = \begin{cases} 1-|\text{t/T}|, & -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$



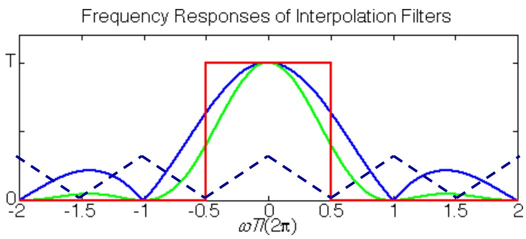
Ideal Interpolation (II)



$$x_r(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n]h_r(t-nT), \quad \text{where} \quad h_r(\text{t}) = \frac{\sin(\pi t/T)}{\pi t/T}$$



Comparison of Interpolation Filters



$$H_0(j\Omega) = T\frac{\sin(\Omega T/2)}{(\Omega T/2)}$$

$$H_1(j\Omega) = T\left(\frac{\sin(\Omega T/2)}{(\Omega T/2)}\right)^2$$

Idealized System (DSP Theory)

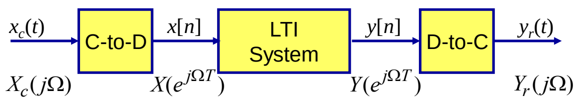
- **A-to-D conversion --> C-to-D conversion**
- **Finite precision arithmetic --> real numbers**
- **D-to-A conversion --> D-to-C conversion**

Summary

A bandlimited signal with highest frequency Ω_N can be reconstructed exactly from samples taken with sampling frequency $\frac{2\pi}{T} \geq 2\Omega_N$

- If an analog signal is not bandlimited, it must be lowpass filtered before sampling in order to avoid distortion by aliasing.
- Filtering specifications can be relaxed by oversampling.

DT Filtering of CT Signals



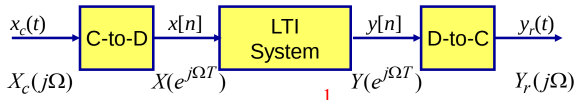
$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega T n} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$Y(e^{j\Omega T}) = H(e^{j\Omega T})X(e^{j\Omega T})$$

$$Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T})$$

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

DT Filtering of CT Signals



$$Y_r(j\Omega) = H(e^{j\Omega T})\left(\cancel{H_r(j\Omega)}^{\,1}\frac{1}{T}\right)\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

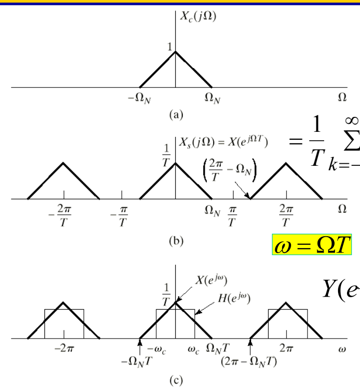
- If the input is bandlimited such that

$$X_c(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

and $2\pi / T \geq 2\Omega_N$, then the overall input and output are related by

$$Y_r(j\Omega) = H(e^{j\Omega T})X_c(j\Omega)$$

D-T Linear Filtering of C-T Signals (I)

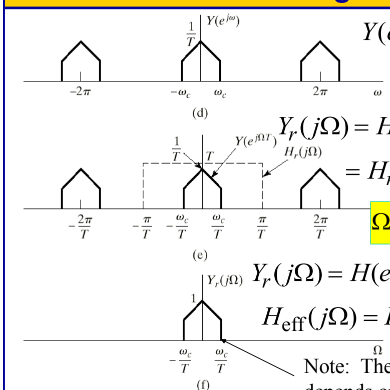


$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega + \frac{2\pi k}{T}\right)\right)$$

$\omega = \Omega T$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

D-T Linear Filtering of C-T Signals (II)



$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T})$$

$$= H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T})$$

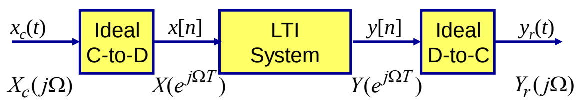
$\Omega = \dfrac{\omega}{T}$

$$Y_r(j\Omega) = H(e^{j\Omega T})H_r(j\Omega)\frac{1}{T}X_c(j\Omega)$$

$$H_{\text{eff}}(j\Omega) = H(e^{j\Omega T}) \quad |\Omega| < \pi / T$$

Note: The effective cutoff frequency depends on the sampling rate!

DT Filtering of CT Signals (III)



$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

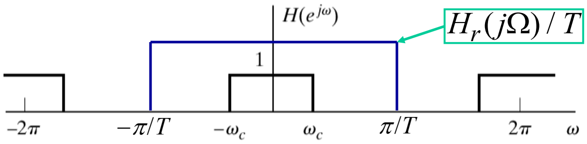
- If the input is bandlimited such that

$$X_c(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

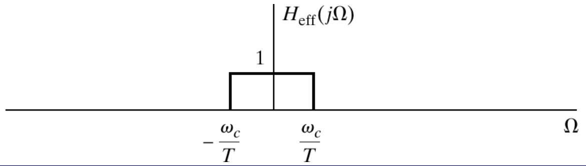
and $2\pi / T \geq 2\Omega_N$, then the overall input and output are related by

$$Y_r(j\Omega) = H(e^{j\Omega T})X_c(j\Omega) = H_{\text{eff}}(j\Omega)X_c(j\Omega)$$

Example



$$H_{\text{eff}}(j\Omega) = H(e^{j\Omega T}) \quad |\Omega| < \frac{\pi}{T}$$



Another Example

- Difference equation:

$$y[n] = ay[n-1] + bx[n]$$

- Frequency response:

$$H(e^{j\omega}) = \frac{b}{1 - ae^{-j\omega}}$$

- Overall frequency response

$$H(j\Omega) = H(e^{j\Omega T}) = \frac{b}{1 - ae^{-j\Omega T}} \quad |\Omega| < \frac{\pi}{T}$$