Georgia Institute of Technology

ECE4270 Fundamentals of DSP

Lecture 13

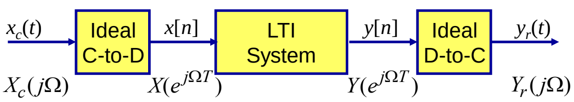
Changing the Sampling Rate Using Digital Filtering

School of ECE
Center for Signal and Information Processing
Georgia Institute of Technology

Overview of Lecture

- DT filtering of CT signals (review)
  - Illustration
  - Examples
- The need to change sampling rates
  - Decimation
  - Interpolation
  - Non-Integer Rate Change
- Over-sampling to ease filtering

DT Filtering of CT Signals (III)



$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

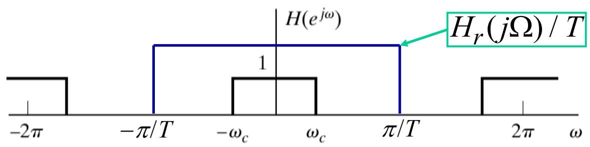
- If the input is bandlimited such that

$$X_c(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

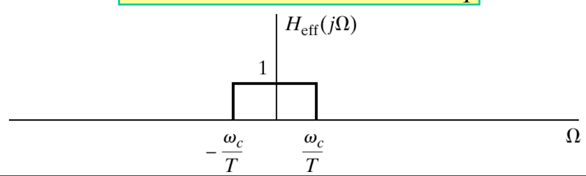
and $2\pi / T \geq 2\Omega_N$, then the overall input and output are related by

$$Y_r(j\Omega) = H(e^{j\Omega T})X_c(j\Omega) = H_{\text{eff}}(j\Omega)X_c(j\Omega)$$

Example



$$H_{\text{eff}}(j\Omega) = H(e^{j\Omega T}) \quad |\Omega| < \frac{\pi}{T}$$



Another Example

- Difference equation:

$$y[n] = ay[n-1] + bx[n]$$

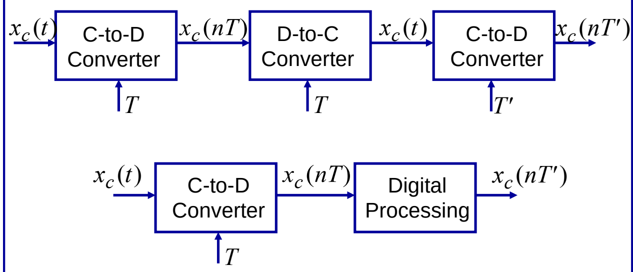
- Frequency response:

$$H(e^{j\omega}) = \frac{b}{1 - ae^{-j\omega}}$$

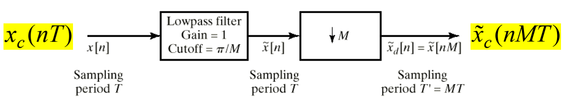
- Overall frequency response

$$H(j\Omega) = H(e^{j\Omega T}) = \frac{b}{1 - ae^{-j\Omega T}} \qquad |\Omega| < \frac{\pi}{T}$$

Sampling Rate Conversion



Decimation - I



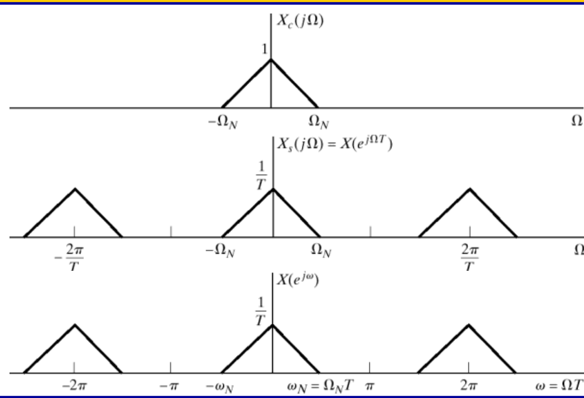
$$x[n] \Leftrightarrow X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$$\tilde{x}[n] \Leftrightarrow \tilde{X}(e^{j\Omega T}) = H(e^{j\Omega T})X(e^{j\Omega T})$$

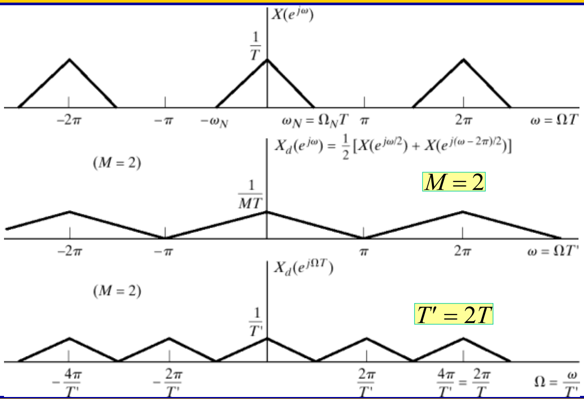
$$\tilde{x}_d[n] \Leftrightarrow \tilde{X}_d(e^{j\Omega MT}) = \frac{1}{MT}\sum_{k=-\infty}^{\infty} \tilde{X}_c\left(j\left(\Omega - \frac{2\pi k}{MT}\right)\right)$$

$$= \frac{1}{M}\sum_{r=0}^{M-1} \tilde{X}(e^{j(\Omega T - 2\pi r)/M})$$

Decimation - II



Decimation - III

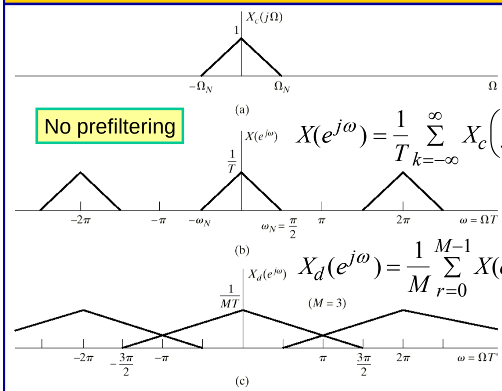


Decimation - IV

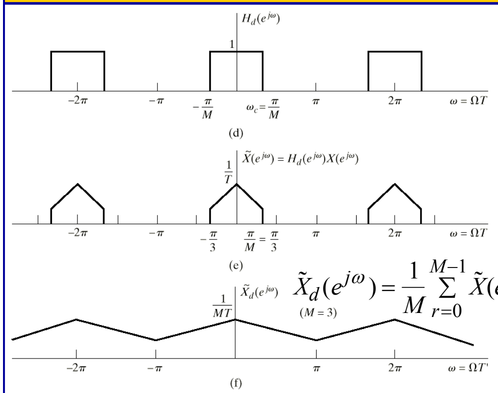
No prefiltering

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{r=0}^{M-1} X(e^{j(\omega-2\pi r)/M})$$

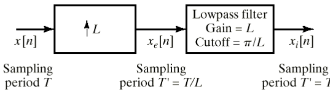


Decimation - V



$$\tilde{X}_d(e^{j\omega}) = \frac{1}{M}\sum_{r=0}^{M-1} \tilde{X}(e^{j(\omega-2\pi r)/M})$$

Interpolation - I

$x_c(nT)$  $x_c(nT/L)$

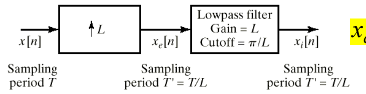
$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL] = \begin{cases} x[n/L], & n = 0, \pm L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin[\pi(n-kL)/L]}{[\pi(n-kL)/L]} \quad \text{or since}$$

$$x_c(t) = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin[\pi(t-kT)/T]}{[\pi(t-kT)/T]} \quad \leftarrow \text{Sampling Theorem}$$

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin[\pi(nT/L-kT)/T]}{[\pi(nT/L-kT)/T]} = x_c(nT/L)$$

Interpolation - II

$x_c(nT)$  $x_c(nT/L)$

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]e^{-j\omega n}$$

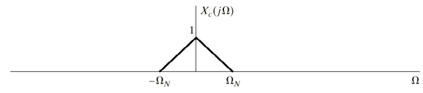
$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega Lk} = X(e^{j\omega L})$$

$$X_e(e^{j\Omega T/L}) = X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$$X_i(e^{j\Omega T/L}) = H_i(e^{j\Omega T/L})X_e(e^{j\Omega T/L})$$

$$= \frac{1}{(T/L)}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T/L}\right)\right)$$

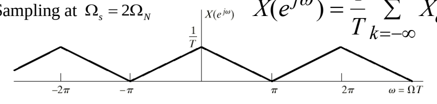
Interpolation - III



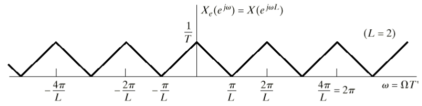
(a)

Sampling at $\Omega_s = 2\Omega_N$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

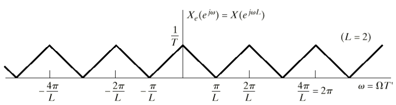


(b)

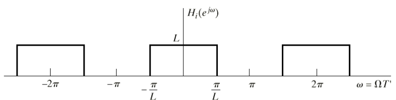


(c)

Interpolation - IV

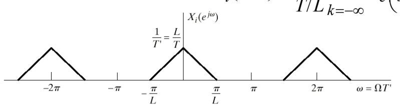


(c)



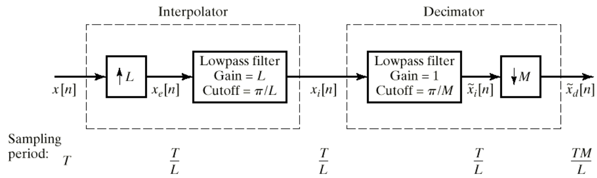
(d)

$$X_i(e^{j\omega}) = \frac{1}{T/L}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T/L} - \frac{2\pi k}{T/L}\right)\right)$$

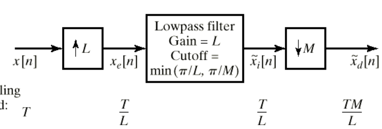


(e)

Non-Integer Rate Change - I

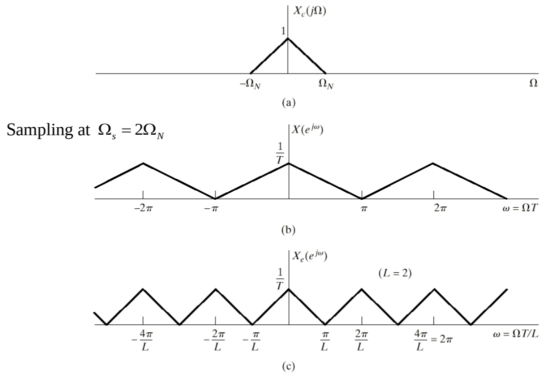


(a)

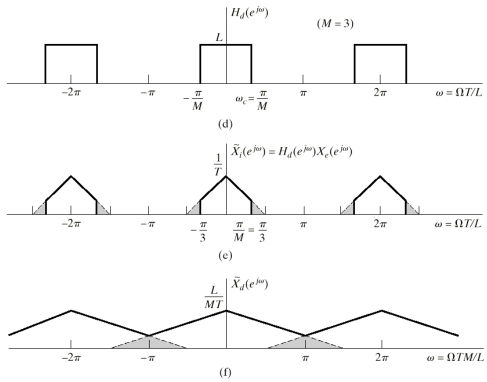


(b)

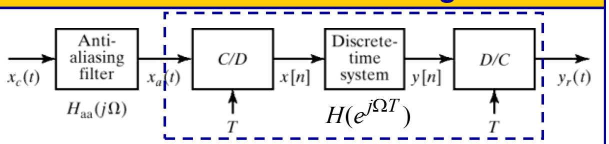
Non-Integer Rate Change - II



Non-Integer Rate Change - III



Anti-Alias Pre-filtering



$Y_r(j\Omega) = H(e^{j\Omega T})X_a(j\Omega)$ if $X_a(j\Omega) = 0$ for $|\Omega| \geq \Omega_N$

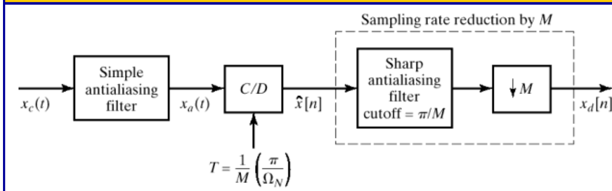
- What is the overall effective frequency response?

$$X_a(j\Omega) = H_{aa}(j\Omega)X_c(j\Omega)$$

$$\Rightarrow \quad X_a(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

$$Y_r(j\Omega) = \underbrace{H(e^{j\Omega T})H_{aa}(j\Omega)}_{H_{\text{eff}}(j\Omega)}X_c(j\Omega)$$

Oversampling Eases Filtering - I

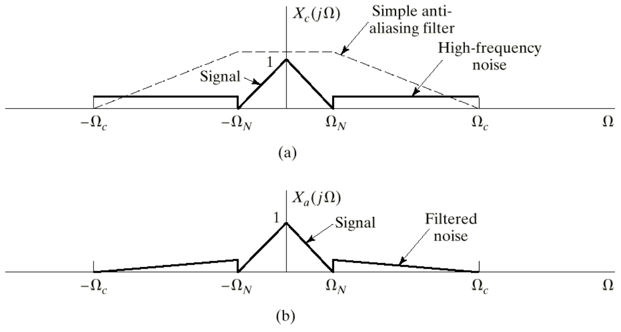


$$X_a(j\Omega) = H_{aa}(j\Omega)X_c(j\Omega)$$

Choose $H_{aa}(j\Omega) = 0$ for $|\Omega| \geq M\Omega_N$

$$\Rightarrow X_a(j\Omega) = 0 \quad \text{for } |\Omega| \geq M\Omega_N$$

Oversampling Eases Filtering - II



Oversampling Eases Filtering - III