Georgia Institute of Technology

CSIP

ECE4270 Fundamentals of DSP

Lecture 15

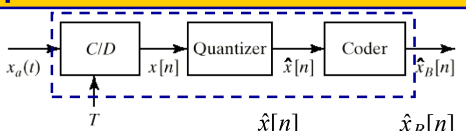
A-to-D conversion

School of ECE
Center for Signal and Information Processing
Georgia Institute of Technology

Overview of Lecture

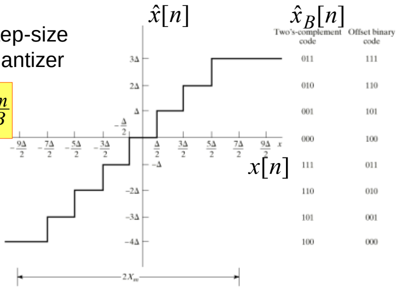
- A-to-D conversion
- Probabilistic analysis of quantization
  - Model
  - Signal-to-noise ratio
- Variation of SNR with Signal Level
- Oversampling can be used to reduce quantization noise
- Introduction to Chapter 5
  - Use of z-transform in analysis of LTI systems
  - Poles and zeros and frequency response

Representation of A-to-D Converter

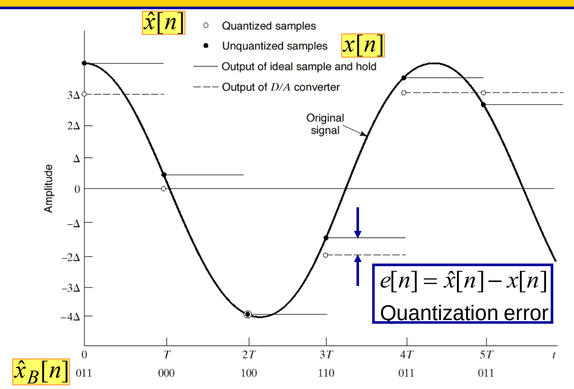


Quantization step-size for (*B*+1)-bit quantizer

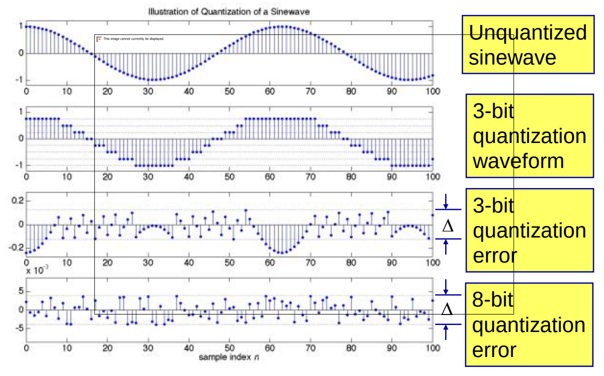
$$\Delta = \frac{2X_m}{2^{B+1}} = \frac{X_m}{2^B}$$



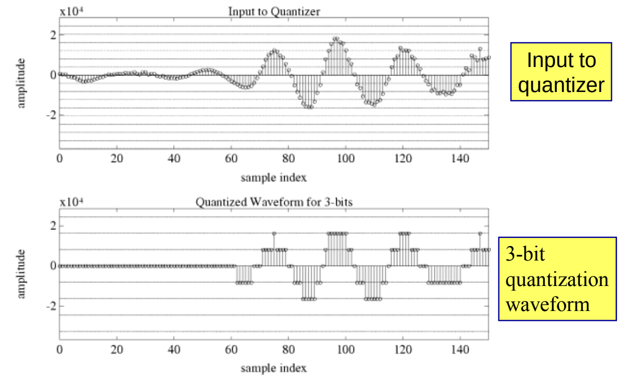
A-to-D and D-to-A Conversion



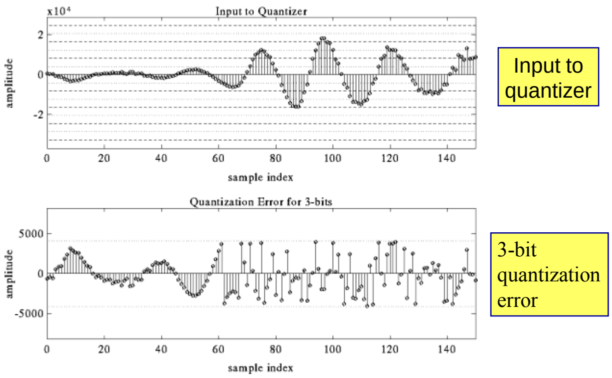
Quantization of a Sine Wave



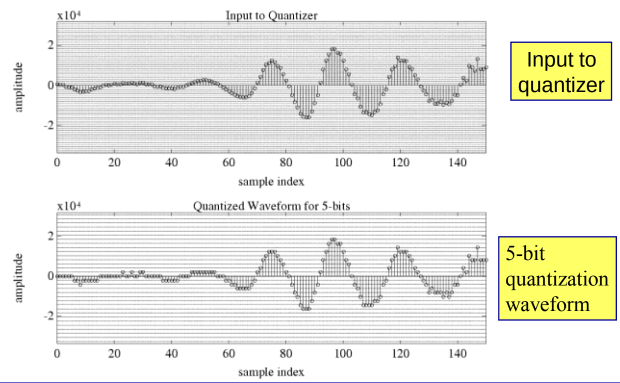
3-Bit Speech Quantization



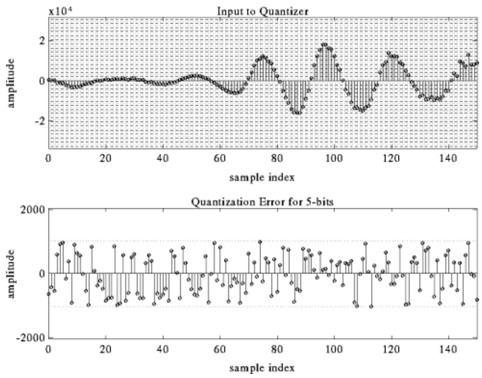
3-Bit Speech Quantization Error



5-Bit Speech Quantization



5-Bit Speech Quantization Error



Input to quantizer

5-bit quantization error

Quantization Error

- Each sample is quantized and each sample has a quantization error defined as

$$e[n] = \hat{x}[n] - x[n]$$

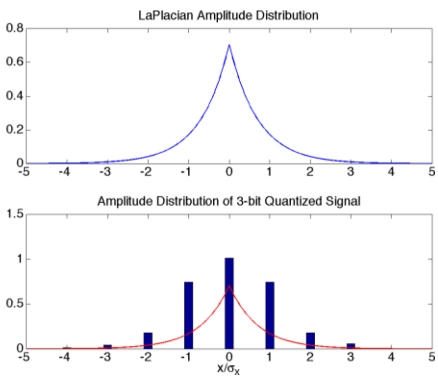
- Since each sample falls in an interval of length Δ, and the quantized sample falls in the middle of that interval,

$$-(\Delta/2) < e[n] \le (\Delta/2).$$

- We call this "quantization noise" because it seems to vary randomly. Clearly, the strength (power) of this noise is proportional to Δ; i.e.,

$$\sigma_e^2 = K\Delta^2$$

Typical Amplitude Distributions



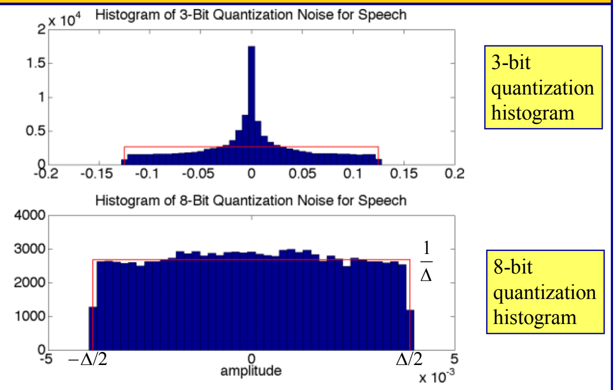
Laplacian distribution is often used as a model for speech signals

A 3-bit quantized signal has only 8 different values

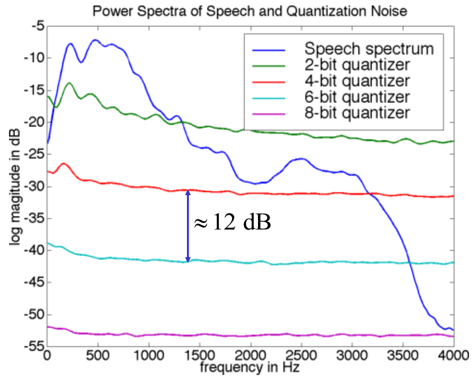
Probabilistic Model for Quantization

- We observed that the quantization error has very complicated variations that suggest a random or noise-like character.
- Random signals are represented by probability distributions and averages such as
  - Mean and mean-square (average power)
  - Histograms
  - Autocorrelation function
  - Power spectrum
- This is a good way to think about quantization noise.

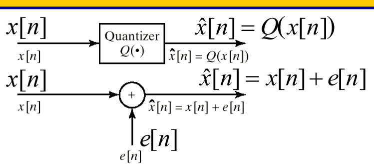
Histograms of Quantization Noise



Spectra of Quantization Noise



Linear Noise Model



- Error is uncorrelated with the input.
- Error is uniformly distributed over the interval

$$-(\Delta/2) < e[n] \le (\Delta/2).$$

- Error is stationary white noise, (i.e. flat spectrum)

$$P_e(\omega) = \sigma_e^2 = \int_{-\Delta/2}^{\Delta/2} \frac{1}{\Delta} e^2 de = \frac{\Delta^2}{12}, \quad |\omega| \le \pi$$

Quantizer Signal-to-Noise Ratio

- Assume $2^{(B+1)}$ levels and amplitude range $2X_m$. Then using a probabilistic analysis we obtain

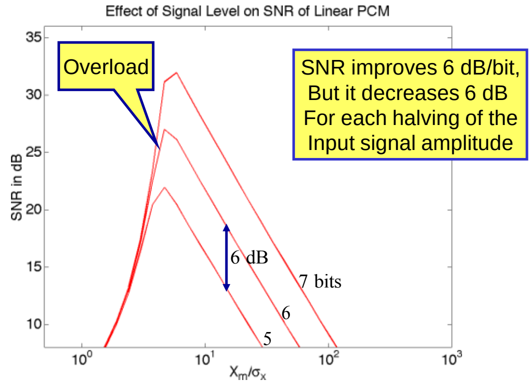
$$\Rightarrow \underbrace{\Delta = \frac{2X_m}{2^{(B+1)}}}_{\text{step size}} = 2^{-B} X_m \quad \Rightarrow \quad \underbrace{\sigma_e^2 = \frac{2^{-2B} X_m^2}{12}}_{\text{noise power}}$$

- Therefore the quantizer SNR is:

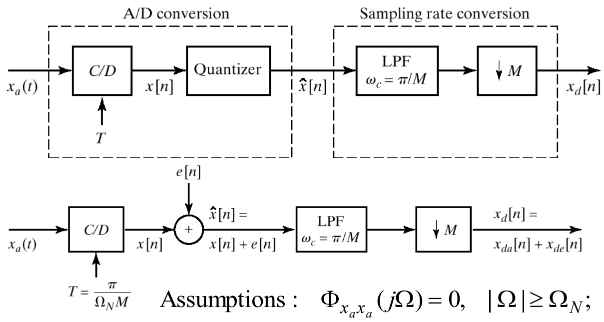
$$\text{SNR} = 10\log_{10}\left(\frac{\sigma_x^2}{\sigma_e^2}\right) = 10\log_{10}\left(\frac{12 \cdot 2^{2B} \sigma_x^2}{X_m^2}\right)$$

$$= 6.02B + 10.8 - 20\log_{10}\left(\frac{X_m}{\sigma_x}\right) \quad \text{(in dB)}$$

Variation of SNR with Signal Level



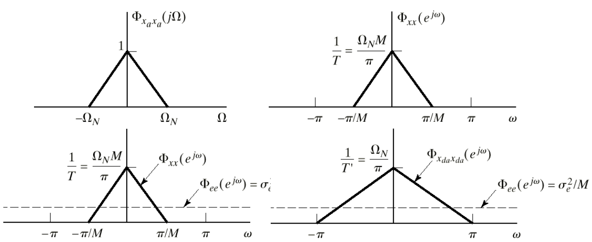
Oversampling in A-to-D Conversion



Assumptions : $\Phi_{x_a x_a}(j\Omega)=0, \quad |\Omega| \geq \Omega_N;$

$$\frac{2\pi}{T}=2M\Omega_N \text{ and } \Phi_{ee}(e^{j\omega})=\sigma_e^2=\frac{\Delta^2}{12}, \quad |\omega|\leq\pi$$

Oversampling Impoves SNR



$$\sigma_{x_{de}}^2=\frac{1}{2\pi}\int_{-\pi/M}^{\pi/M}\sigma_e^2 d\omega=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sigma_e^2}{M}\sigma_e^2 d\omega=\frac{\sigma_e^2}{M}$$

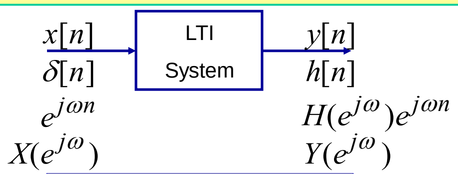
$$\text{SNR}=6.02B+10.8+10\log_{10}(M)-20\log_{10}\left(\frac{X_m}{\sigma_x}\right)$$

Summary

- Quantization of signal values and results of computation is unavoidable in a digital system.
- We can analyze quantization error using a random noise model.
- The more bits in the number representation, the lower the noise. A soft stated theorem is that “the signal-to-noise ratio increases 6 dB with each added bit”; however, remember that if the signal level decreases while keeping the quantizer step-size the same it is like throwing away bits!

Convolution Theorem

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]\Leftrightarrow Y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})$$

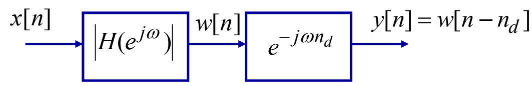


$$Y(e^{j\omega})=H(e^{j\omega})X(e^{j\omega})$$

$$|Y(e^{j\omega})|=|H(e^{j\omega})||X(e^{j\omega})|$$

$$\angle Y(e^{j\omega})=\angle H(e^{j\omega})+\angle X(e^{j\omega})$$

Linear Phase Means Delay



- Since we can consider the effects of magnitude and phase separately, it follows that linear phase of the form $\angle H(e^{j\omega})=-\omega n_d$ implies delay of n_d samples.
- Thus an ideal lowpass filter with delay has

$$h_{\mathrm{lp}}[n]=\frac{\sin\omega_c(n-n_d)}{\pi(n-n_d)}\Leftrightarrow H(e^{j\omega})=\begin{cases}e^{-j\omega n_d} & |\omega|<\omega_c\\ 0 & \omega_c<|\omega|<\pi\end{cases}$$

Frequency Response Functions

$$H(e^{j\omega})=|H(e^{j\omega})|e^{j\angle H(e^{j\omega})}$$

- Log-magnitude (in dB)

$$20\log_{10}|H(e^{j\omega})|$$

- Phase (in radians)

$$\angle H(e^{j\omega})=\arg\left[H(e^{j\omega})\right]$$

- Group delay (in samples)

$$\tau(\omega)=\mathrm{grd}\left[H(e^{j\omega})\right]=-\frac{d}{d\omega}\left\{\angle H(e^{j\omega})\right\}$$

Rational System Functions

- Consider a general difference equation of the form

$$\sum_{k=0}^{N}a_k y[n-k]=\sum_{k=0}^{M}b_k x[n-k]$$

- Rational system function of a causal and stable LTI system

$$H(z)=\frac{\sum_{k=0}^{M}b_k z^{-k}}{\sum_{k=0}^{N}a_k z^{-k}}=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

Causal $\Rightarrow$ $|z|>\max_k|d_k|$

Stable $\Rightarrow$ $\max_k|d_k|<1$

Impulse Response

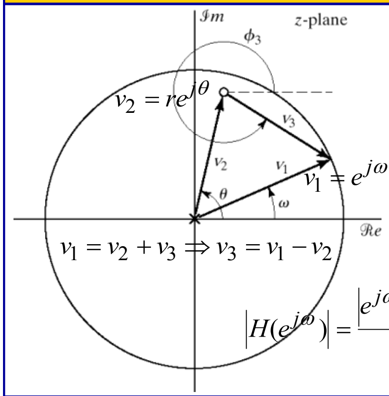
- We can make a partial fraction expansion of the rational system function:

$$H(z)=\underbrace{\left[\sum_{r=0}^{(M-N)} B_r z^{-r}\right]}_{\text{if } M\geq N}+\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}} \qquad |z|>\max_k |d_k|$$

- The inverse z-transform gives the impulse response

$$h[n]=\underbrace{\left[\sum_{r=0}^{(M-N)} B_r \delta[n-r]\right]}_{\text{if } M\geq N}+\sum_{k=1}^{N} A_k d_k^n u[n]$$

Pole-Zero Plot



$$H(z)=(1-re^{j\theta}z^{-1})$$

$$=\frac{z-re^{j\theta}}{z}$$

$$H(e^{j\omega})=\frac{e^{j\omega}-re^{j\theta}}{e^{j\omega}}$$

$$=\frac{v_3}{v_1}$$

$$\left|H(e^{j\omega})\right|=\frac{\left|e^{j\omega}-re^{j\theta}\right|}{\left|e^{j\omega}\right|}=\frac{|v_3|}{|v_1|}=|v_3|$$

Example (I)

- System function:

$$H(z)=\frac{1}{\left(1-re^{j\theta}z^{-1}\right)\left(1-re^{-j\theta}z^{-1}\right)}=\frac{1}{1-2r\cos\theta z^{-1}+r^2z^{-2}}$$

- Difference equation:

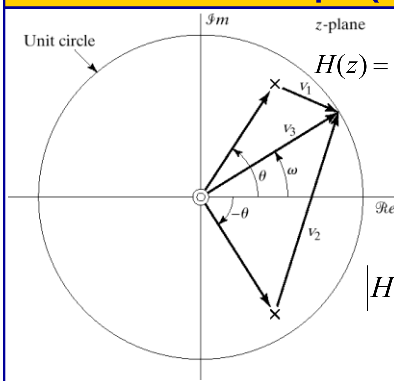
$$y[n]=2r\cos\theta y[n-1]-r^2y[n-2]+x[n]$$

- Impulse response:

$$H(z)=\frac{\frac{1}{1-e^{-j2\theta}}}{\left(1-re^{j\theta}z^{-1}\right)}+\frac{\frac{1}{1-e^{j2\theta}}}{\left(1-re^{-j\theta}z^{-1}\right)}$$

$$h[n]=\frac{r^n\sin[\theta(n+1)]}{\sin\theta}u[n]$$

Example (II)



$$H(z)=\frac{z^2}{\left(z-re^{j\theta}\right)\left(z-re^{-j\theta}\right)}$$

$$\left|H(e^{j\omega})\right|=\frac{|v_3|^2}{|v_1|\cdot|v_2|}$$

Example (III): Magnitude