ECE4270
Fundamentals of DSP

Lecture 16

Properties of LTI Systems

School of ECE
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Overview of Lecture

- Use of z-transform in analysis of LTI systems
- Poles and zeros and frequency response
- Minimum-phase and allpass systems

Frequency Response Functions

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$

- Log-magnitude (in dB)

$$20\log_{10}|H(e^{j\omega})|$$

- Phase (in radians)

$$\angle H(e^{j\omega}) = \arg\left[H(e^{j\omega})\right]$$

- Group delay (in samples)

$$\tau(\omega) = \text{grd}\left[H(e^{j\omega})\right] = -\frac{d}{d\omega}\left\{\angle H(e^{j\omega})\right\}$$

Rational System Functions

- Consider a general difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

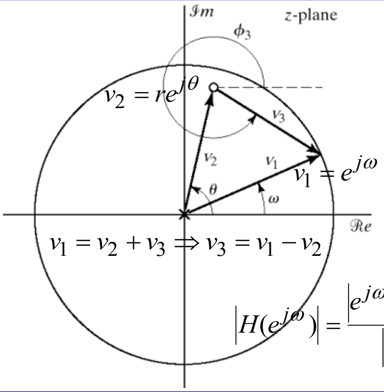
- Rational system function of a causal and stable LTI system

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

Causal $\Rightarrow$ $|z| > \max_k |d_k|$

Stable $\Rightarrow$ $\max_k |d_k| < 1$

Pole-Zero Plot



$$H(z) = (1 - re^{j\theta} z^{-1}) = \frac{z - re^{j\theta}}{z}$$

$$H(e^{j\omega}) = \frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}} = \frac{v_3}{v_1}$$

$$\left|H(e^{j\omega})\right| = \frac{\left|e^{j\omega} - re^{j\theta}\right|}{\left|e^{j\omega}\right|} = \frac{|v_3|}{|v_1|} = |v_3|$$

Example (I)

- System function:

$$H(z) = \frac{1}{\left(1 - re^{j\theta} z^{-1}\right)\left(1 - re^{-j\theta} z^{-1}\right)} = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

- Difference equation:

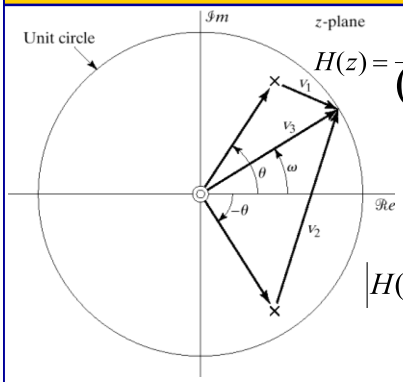
$$y[n] = 2r\cos\theta y[n-1] - r^2 y[n-2] + x[n]$$

- Impulse response:

$$H(z) = \frac{\dfrac{1}{1 - e^{-j2\theta}}}{\left(1 - re^{j\theta} z^{-1}\right)} + \frac{\dfrac{1}{1 - e^{j2\theta}}}{\left(1 - re^{-j\theta} z^{-1}\right)}$$

$$h[n] = \frac{r^n \sin[\theta(n+1)]}{\sin\theta} u[n]$$

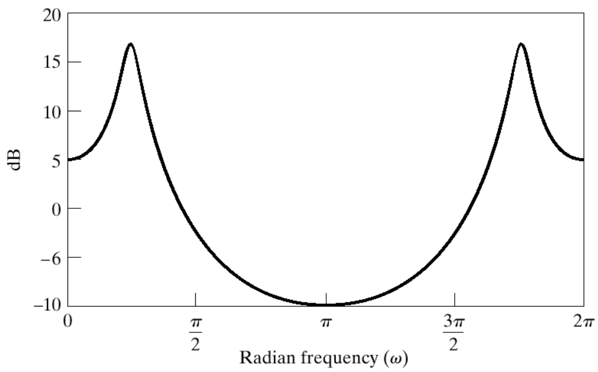
Example (II)



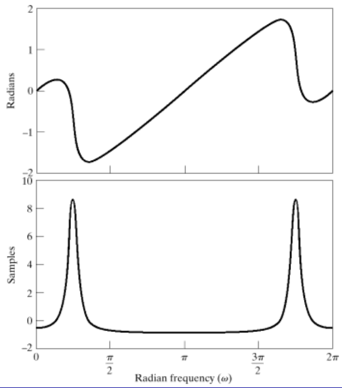
$$H(z) = \frac{z^2}{\left(z - re^{j\theta}\right)\left(z - re^{-j\theta}\right)}$$

$$\left|H(e^{j\omega})\right| = \frac{|v_3|^2}{|v_1| \cdot |v_2|}$$

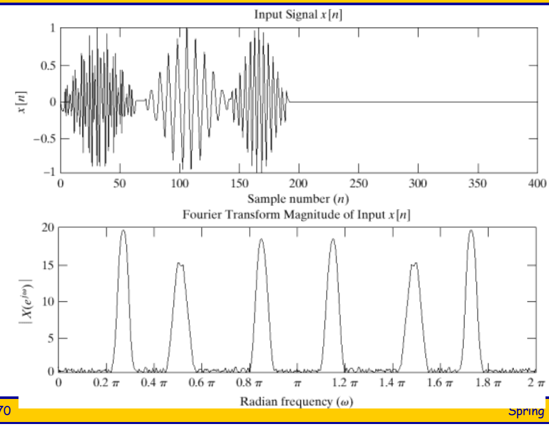
Example (III): Magnitude



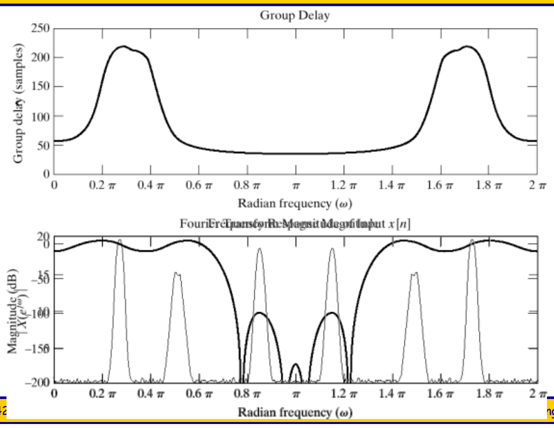
Example (IV): Phase and Group Delay



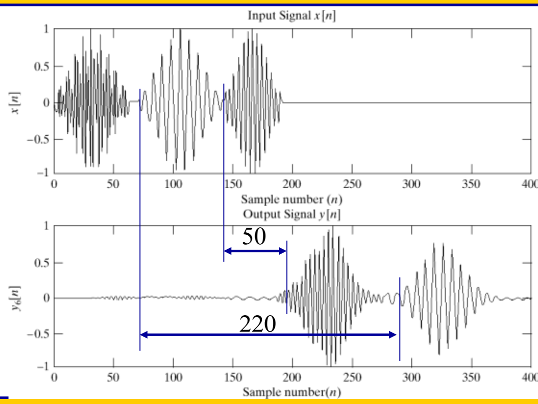
Group delay (I): Input Signal



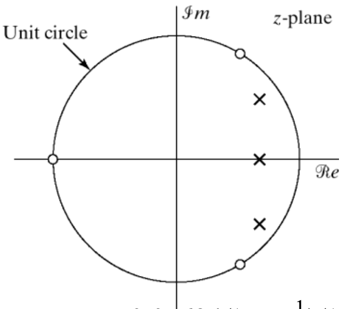
Group delay (II): Frequency Response of Filter



Group delay (III): Output Signal



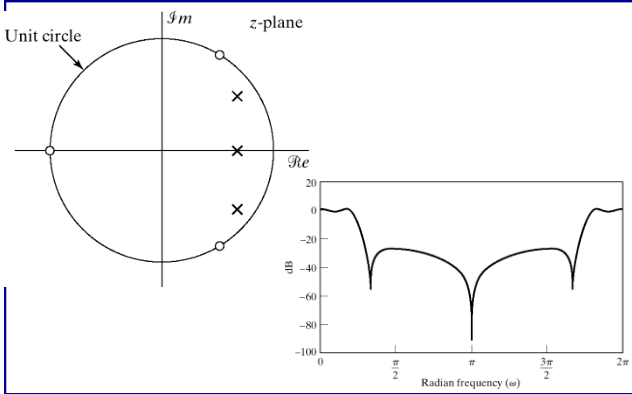
Example of an IIR Filter (I)



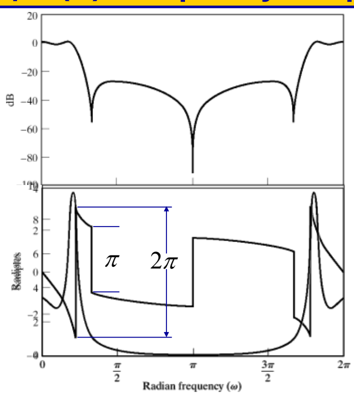
What can you say about the impulse response of this System?

$$H(z)=\frac{0.05634(1+z^{-1})(1-1.0166z^{-1}+z^{-2})}{(1-0.683z^{-1})(1-1.4461z^{-1}+0.7957z^{-2})}$$

Example (II): Frequency Response



Example (III): Frequency Response



Allpass Systems (I)

- An allpass system has frequency response

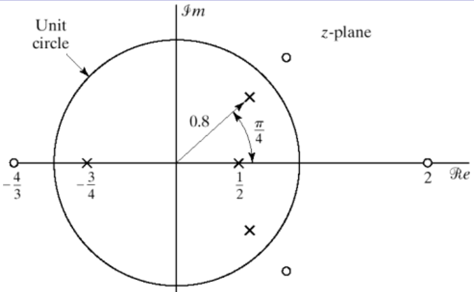
$$\left|H(e^{j\omega})\right| = A = \text{constant} \quad \forall\omega$$

- The general allpass system has system function

$$H_{\text{ap}}(z) = A\prod_{k=1}^{M_r}\frac{z^{-1}-d_k}{1-d_kz^{-1}}\prod_{k=1}^{M_c}\frac{(z^{-1}-e_k^*)(z^{-1}-e_k)}{(1-e_kz^{-1})(1-e_k^*z^{-1})}$$

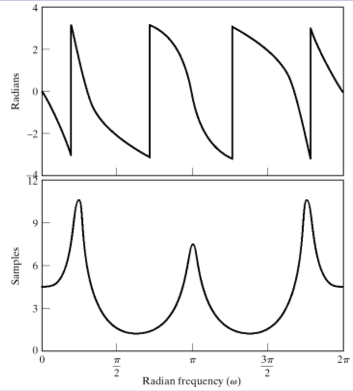
- For every pole inside the unit circle there is a zero at the conjugate reciprocal location.

Allpass Systems (II): Example



$$H_{\text{ap}}(z)=\frac{\left(z^{-1}+0.75\right)\left(z^{-1}-0.5\right)\left(z^{-1}-0.8e^{-j\pi/4}\right)\left(z^{-1}-0.8e^{j\pi/4}\right)}{\left(1+0.75z^{-1}\right)\left(1-0.5z^{-1}\right)\left(1-0.8e^{j\pi/4}z^{-1}\right)\left(1-0.8e^{-j\pi/4}z^{-1}\right)}$$

Allpass Systems (III) Phase and Group Delay



Note that "unwrapped phase" would always be negative for an allpass system.

$$\arg\left[H_{\text{ap}}(e^{j\omega})\right]\le 0$$

Note that group delay would always be positive for an allpass system.

$$\text{grd}\left[H_{\text{ap}}(e^{j\omega})\right]\ge 0$$

Minimum-Phase Systems

- A minimum-phase system is a causal LTI system whose poles and zeros are inside the unit circle.

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}\qquad |c_k|<1 \quad |d_k|<1$$

- The inverse filter for H(z) would be causal and stable.

$$H_i(z)=\frac{1}{H(z)}==\frac{a_0}{b_0}\frac{\prod_{k=1}^{M}(1-d_k z^{-1})}{\prod_{k=1}^{N}(1-c_k z^{-1})}$$

Minimum-Phase/Allpass Representation

- Any LTI system can be represented as a cascade of a minimum-phase system with an allpass system.

$$H(z)=H_{\min}(z)H_{\text{ap}}(z)$$

- Suppose *H*(*z*) has a single zero outside the unit circle at $z=1/c^*$. Then

$$H(z)=H_1(z)\left(z^{-1}-c^*\right)=H_1(z)\left(1-cz^{-1}\right)\frac{\left(z^{-1}-c^*\right)}{\left(1-cz^{-1}\right)}$$

$$H_{\min}(z)=H_1(z)\left(1-cz^{-1}\right)$$

- Repeat for all the zeros outside the unit circle.

More Properties

$$H(z) = H_{\min}(z)H_{ap}(z) \Rightarrow \text{a family of systems}$$

$$\left|H(e^{j\omega})\right| = \left|H_{\min}(e^{j\omega})\right|\left|H_{ap}(e^{j\omega})\right| = \left|H_{\min}(e^{j\omega})\right|$$

- Minimum phase lag:

$$-\arg\left[H(e^{j\omega})\right] = -\arg\left[H_{\min}(e^{j\omega})\right] - \arg\left[H_{ap}(e^{j\omega})\right]$$

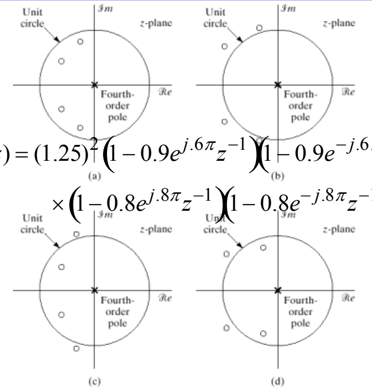
$$\arg\left[H_{ap}(e^{j\omega})\right] \le 0 \Rightarrow \text{ phase lag increased by allpass}$$

- Minimum group delay

$$\text{grd}\left[H(e^{j\omega})\right] = \text{grd}\left[H_{\min}(e^{j\omega})\right] + \text{grd}\left[H_{ap}(e^{j\omega})\right]$$

$$\text{grd}\left[H_{ap}(e^{j\omega})\right] \ge 0 \Rightarrow \text{ grd increased by allpass}$$

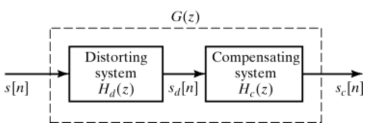
Example



$$H_{\min}(z) = (1.25)^2\left(1 - 0.9e^{j.6\pi}z^{-1}\right)\left(1 - 0.9e^{-j.6\pi}z^{-1}\right)$$
$$\times\left(1 - 0.8e^{j.8\pi}z^{-1}\right)\left(1 - 0.8e^{-j.8\pi}z^{-1}\right)$$

Minimum-Phase/Allpass Representation

- Compensation of non-minimum phase systems



$$H_d(z) = H_{d\min}(z)H_{ap}(z)$$

$$H_c(z) = \frac{1}{H_{d\min}(z)} \quad \Rightarrow \quad G(z) = H_{ap}(z)$$