ECE4270 Fundamentals of DSP

Lecture 17

Properties of LTI Systems (II)

School of ECE
Center for Signal and Information Processing
Georgia Institute of Technology

Overview of Lecture

- Minimum-phase and allpass systems
- Linear phase systems
- Generalized linear phase systems
  - Type I
  - Type II
  - Type II
  - Type IV

Allpass Systems (I)

- An allpass system has frequency response

$$\left|H(e^{j\omega})\right| = A = \text{constant} \quad \forall \omega$$

- The general allpass system has system function

$$H_{\text{ap}}(z) = A\prod_{k=1}^{M_r}\frac{z^{-1}-d_k}{1-d_k z^{-1}}\prod_{k=1}^{M_c}\frac{(z^{-1}-e_k^*)(z^{-1}-e_k)}{(1-e_k z^{-1})(1-e_k^* z^{-1})}$$

- For every pole inside the unit circle there is a zero at the conjugate reciprocal location.

Minimum-Phase Systems

- A minimum-phase system is a causal LTI system whose poles and zeros are inside the unit circle.

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \qquad |c_k|<1 \quad |d_k|<1$$

- The inverse filter for H(z) would be causal and stable.

$$H_i(z) = \frac{1}{H(z)} == \frac{a_0}{b_0}\frac{\prod_{k=1}^{M}(1-d_k z^{-1})}{\prod_{k=1}^{N}(1-c_k z^{-1})}$$

Minimum-Phase/Allpass Representation

- Any LTI system can be represented as a cascade of a minimum-phase system with an allpass system.

$$H(z) = H_{\min}(z) H_{ap}(z)$$

- Suppose *H*(*z*) has a single zero outside the unit circle at $z = 1/c^*$. Then

$$H(z) = H_1(z)\left(z^{-1} - c^*\right) = H_1(z)\left(1 - cz^{-1}\right)\frac{\left(z^{-1} - c^*\right)}{\left(1 - cz^{-1}\right)}$$

$$H_{\min}(z) = H_1(z)\left(1 - cz^{-1}\right)$$

- Repeat for all the zeros outside the unit circle.

More Properties

$$H(z) = H_{\min}(z) H_{ap}(z) \Rightarrow \text{a family of systems}$$

$$\left|H(e^{j\omega})\right| = \left|H_{\min}(e^{j\omega})\right|\left|H_{ap}(e^{j\omega})\right| = \left|H_{\min}(e^{j\omega})\right|$$

- Minimum phase lag:

$$-\arg\left[H(e^{j\omega})\right] = -\arg\left[H_{\min}(e^{j\omega})\right] - \arg\left[H_{ap}(e^{j\omega})\right]$$

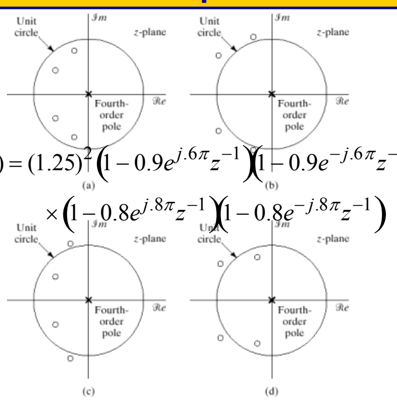
$$\arg\left[H_{ap}(e^{j\omega})\right] \le 0 \Rightarrow \text{ phase lag increased by allpass}$$

- Minimum group delay

$$\text{grd}\left[H(e^{j\omega})\right] = \text{grd}\left[H_{\min}(e^{j\omega})\right] + \text{grd}\left[H_{ap}(e^{j\omega})\right]$$

$$\text{grd}\left[H_{ap}(e^{j\omega})\right] \ge 0 \Rightarrow \text{ grd increased by allpass}$$

Example

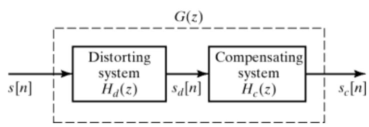


$$H_{\min}(z) = (1.25)^2\left(1 - 0.9e^{j.6\pi}z^{-1}\right)\left(1 - 0.9e^{-j.6\pi}z^{-1}\right)$$

$$\times\left(1 - 0.8e^{j.8\pi}z^{-1}\right)\left(1 - 0.8e^{-j.8\pi}z^{-1}\right)$$

Minimum-Phase/Allpass Representation

- Compensation of non-minimum phase systems



$$H_d(z) = H_{d\min}(z) H_{ap}(z)$$

$$H_c(z) = \frac{1}{H_{d\min}(z)} \quad \Rightarrow \quad G(z) = H_{ap}(z)$$

Ideal Lowpass with Delay

- Frequency response:

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

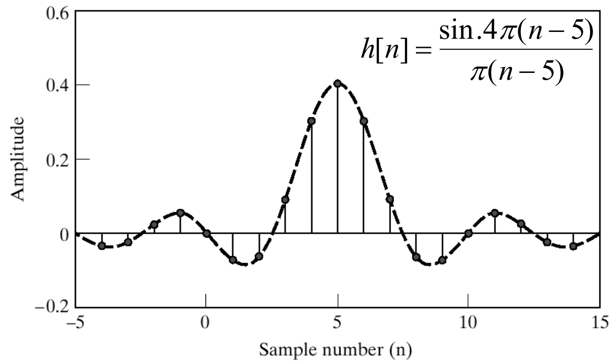
- Magnitude and phase:

$$\left|H(e^{j\omega})\right| = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases} \qquad \angle H(e^{j\omega}) = -\omega\alpha$$

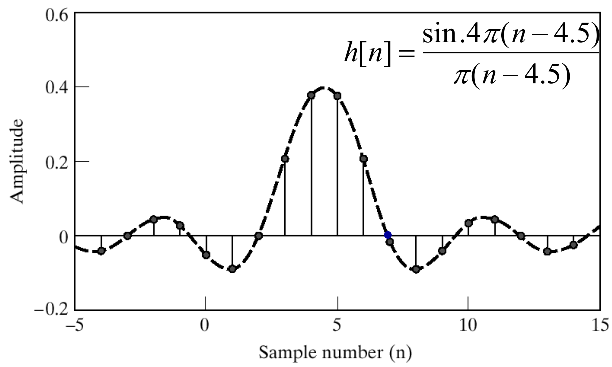
- Impulse response:

$$h[n] = \frac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)} \qquad -\infty < n < \infty$$

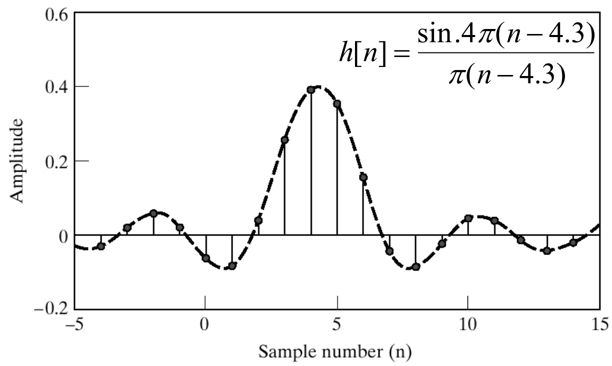
Ideal Lowpass with Linear Phase



Ideal Lowpass with Linear Phase

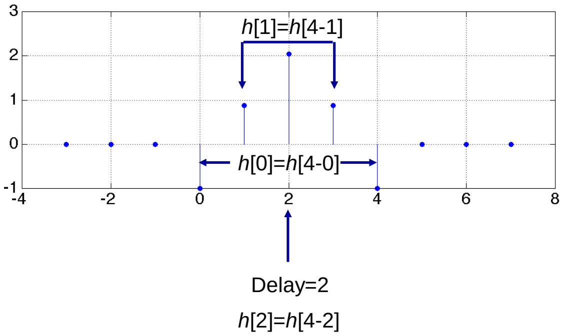


Ideal Lowpass with Linear Phase



Impulse Response-I

Impulse Response: $h[n]=-\delta[n]+.88\delta[n-1]+2.04\delta[n-2]+.88\delta[n-3]-\delta[n-4]$



Pole and Zero Locations-II

$$H(z)=-1+.88z^{-1}+2.04z^{-2}+.88z^{-3}-z^{-4}$$

$$H(z)=\frac{-z^{4}+.88z^{3}+2.04z^{2}+.88z-1}{z^{4}}$$

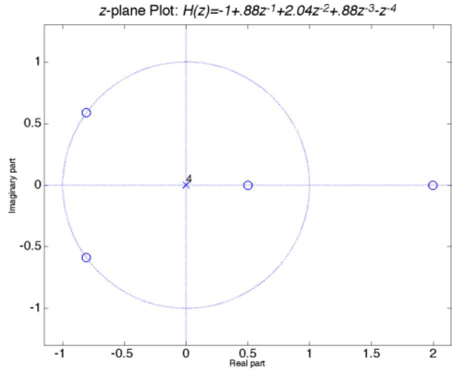
$$H(z^{-1})=-z^{4}+.88z^{3}+2.04z^{2}+.88z-1$$

$$H(z^{-1})=z^{4}H(z)$$

$$H(z_0)=0 \quad \Leftrightarrow \quad H(z_0^{-1})=0$$

Pole-Zero Plot-III

z-plane Plot: $H(z)=-1+.88z^{-1}+2.04z^{-2}+.88z^{-3}-z^{-4}$



Frequency Response-IV

$$H(e^{j\omega})=-1+.88e^{-j\omega}+2.04e^{-j\omega 2}$$
$$+.88e^{-j\omega 3}-e^{-j\omega 4}$$

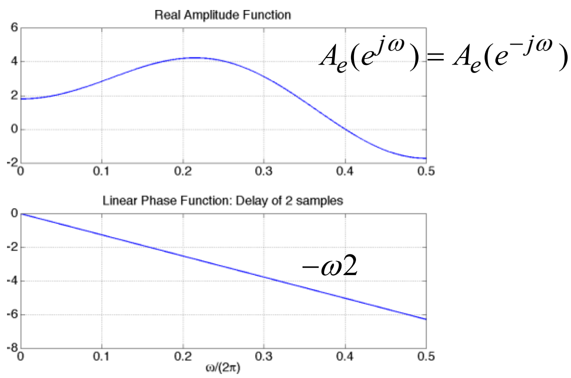
$$H(e^{j\omega})=\begin{pmatrix}-e^{j\omega 2}+.88e^{j\omega}+2.04 \\ +.88e^{-j\omega}-e^{-j\omega 2}\end{pmatrix}e^{-j\omega 2}$$

$$H(e^{j\omega})=\begin{pmatrix}2.04+.88(e^{j\omega}+e^{-j\omega}) \\ -(e^{j\omega 2}+e^{-j\omega 2})\end{pmatrix}e^{-j\omega 2}$$

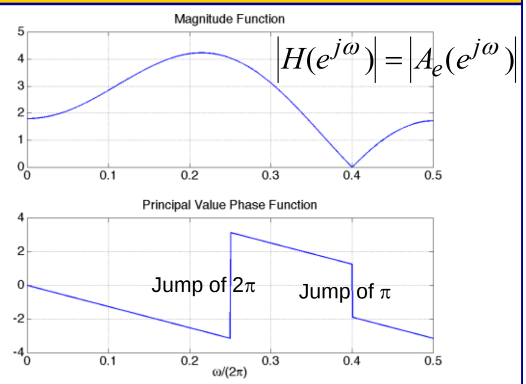
$$H(e^{j\omega})=[2.04+1.76\cos\omega-2\cos(2\omega)]e^{-j\omega 2}$$

$$H(e^{j\omega})=A_e(e^{j\omega})e^{-j\omega 2}, \quad A_e(e^{j\omega})=A_e(e^{-j\omega})$$

Amplitude and Angle Plot-V



Magnitude and Phase Plot-VI

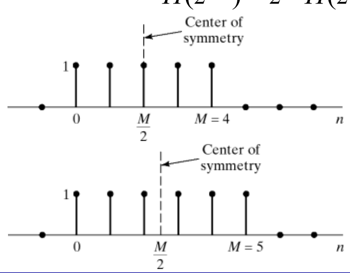


Generalized Linear Phase FIR Systems

- Types I & II: $h[M-n] = h[n] \quad 0 \le n \le M$

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}, \quad A_e(e^{j\omega}) = A_e(e^{-j\omega})$$

$$H(z^{-1}) = z^M H(z)$$



Type I:
M even
integer delay

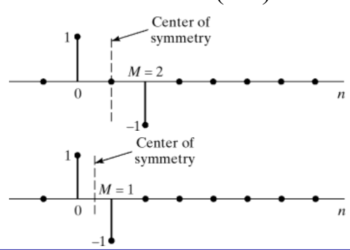
Type II:
M odd
half sample
delay

Generalized Linear Phase FIR Systems

- Types III & IV: $h[M-n] = -h[n] \quad 0 \le n \le M$

$$H(e^{j\omega}) = jA_o(e^{j\omega})e^{-j\omega M/2}, \quad A_o(e^{j\omega}) = -A_o(e^{-j\omega})$$

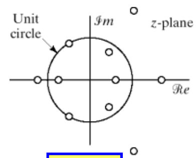
$$H(z^{-1}) = -z^M H(z)$$



Type III:
M even
integer delay

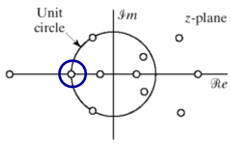
Type IV:
M odd
half sample
delay

Zero Locations for FIR Linear Phase

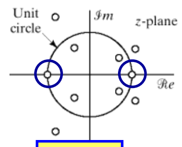


Type I

$$H(z^{-1}) = z^{M} H(z)$$

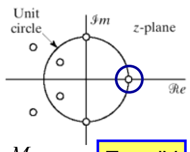


Type II



Type III

$$H(z^{-1}) = -z^{M} H(z)$$



Type IV