ECE4270
Fundamentals of DSP

Lecture 18

Implementation of LTI Systems

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## Overview of Lecture

- Chapter 6
  - Signal flow graphs and block diagrams
    - IIR systems
      - Direct forms
      - Cascade form
      - Parallel form
    - FIR systems Implementations
- Two's-complement arithmetic
  - Integers and fractions
  - Scaling for fixed-point arithmetic



## Digital Filters

- General *N*th-order difference equation:

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

- System function:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = A \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

- There is a direct correspondence between the difference equation and the system function when the numerator and denominator are written as polynomials in $z^{-1}$.



## Direct Form I Implementation

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \underbrace{\left( \frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right)}_{\text{poles}} \underbrace{\sum_{k=0}^{M} b_k z^{-k}}_{\text{zeros}}$$

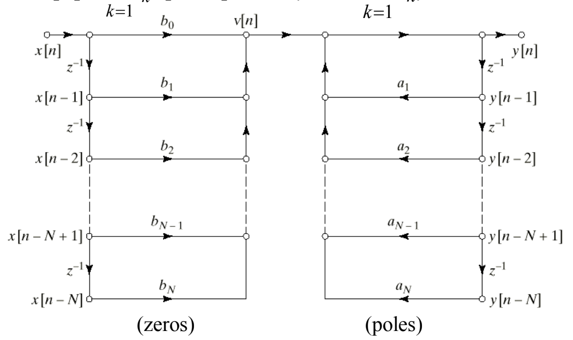
$$Y(z) = H(z)X(z)$$

$$v[n] = \sum_{k=0}^{M} b_k x[n-k] \quad \text{(zeros)}$$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + v[n] \quad \text{(poles)}$$

## Signal Flow Graph IIR Direct Form I

$$v[n]=\sum_{k=1}^{M} b_k x[n-k] \qquad y[n]=\sum_{k=1}^{N} a_k y[n-k]+v[n]$$






## Direct Form II Implementation

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}}=\underbrace{\sum_{k=0}^{M} b_k z^{-k}}_{\text{zeros}}\underbrace{\left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right)}_{\text{poles}}$$

$$Y(z)=H(z)X(z)$$

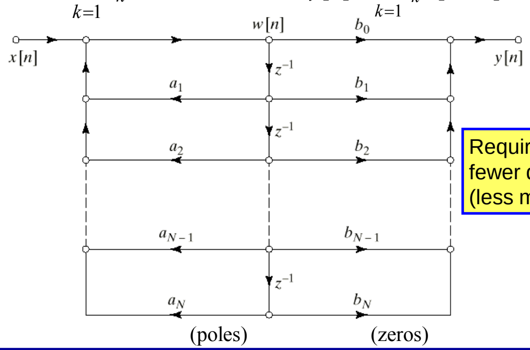
$$w[n]=\sum_{k=1}^{N} a_k w[n-k]+x[n] \quad \text{(poles)}$$

$$y[n]=\sum_{k=0}^{M} b_k w[n-k] \quad \text{(zeros)}$$



## IIR Direct Form II

$$w[n]=\sum_{k=1}^{N} a_k w[n-k]+x[n] \qquad y[n]=\sum_{k=1}^{M} b_k w[n-k]$$




Requires fewer delays (less memory)



## Cascade Form

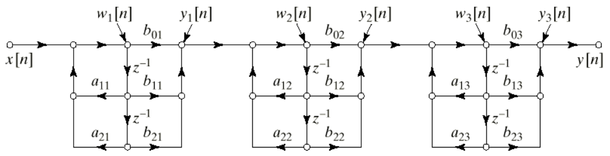
- Express H(z) in terms of poles and zeros

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}}=A\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

- Group poles and zeros as second-order factors

$$H(z)=\prod_{k=1}^{N_s}\frac{b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2}}{1-a_{1k}z^{-1}+a_{2k}z^{-2}}$$

## IIR Cascade Form




$$H(z)=\prod_{k=1}^{N_s}\frac{b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2}}{1-a_{1k}z^{-1}+a_{2k}z^{-2}}$$

$$y_0[n]=x[n],$$

$$w_k[n]=a_{1k}w_k[n-1]+a_{2k}w_k[n-2]+y_{k-1}[n],\quad k=1,2,\ldots,N_s,$$

$$y_k[n]=b_{0k}w_k[n]+b_{1k}w_k[n-1]+b_{2k}w_k[n-2],\quad k=1,2,\ldots,N_s,$$

$$y[n]=y_{N_s}[n].$$



## Parallel Form

- Make a partial fraction expansion of H(z)

$$H(z)=\sum_{k=0}^{N_p}C_kz^{-k}+\sum_{k=1}^{N_1}\frac{B_k}{1-c_kz^{-1}}$$

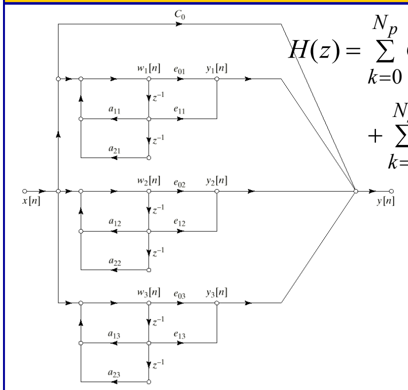
$$+\sum_{k=1}^{N_2}\left(\frac{A_k}{1-d_kz^{-1}}+\frac{A_k^*}{1-d_k^*z^{-1}}\right)$$

- Group terms in second order factors

$$H(z)=\sum_{k=0}^{N_p}C_kz^{-k}+\sum_{k=1}^{N_s}\frac{e_{0k}+e_{1k}z^{-1}}{1-a_{1k}z^{-1}-a_{2k}z^{-2}}$$



## IIR Parallel Form

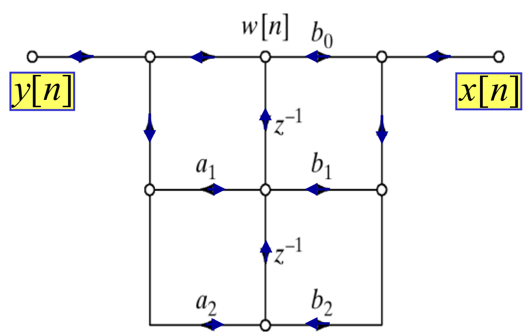



$$H(z)=\sum_{k=0}^{N_p}C_kz^{-k}$$

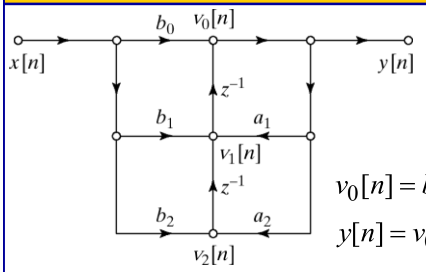
$$+\sum_{k=1}^{N_s}\frac{e_{0k}+e_{1k}z^{-1}}{1-a_{1k}z^{-1}-a_{2k}z^{-2}}$$



## Transposed Forms



## Transposed Form Difference Equations




$$v_0[n] = b_0 x[n] + v_1[n-1]$$

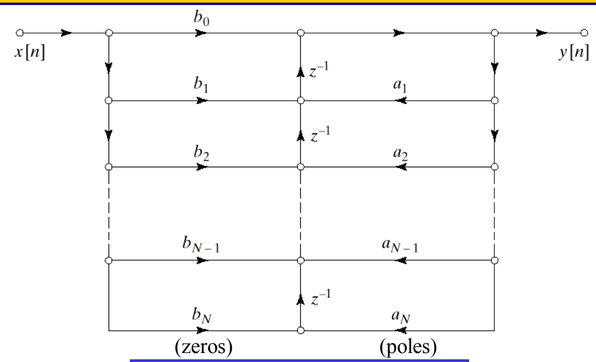
$$y[n] = v_0[n]$$

$$v_1[n] = b_1 x[n] + v_2[n-1] + a_1 v_0[n]$$

$$v_2[n] = b_2 x[n] + a_2 v_0[n]$$



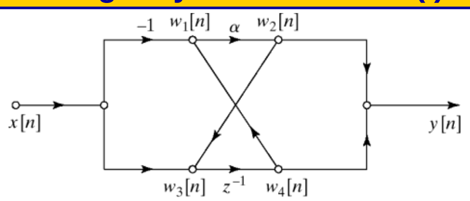
## Transposed IIR




Used in MATLAB's filter( ) function



## Finding a System Function (I)




$$w_1[n] = w_4[n] - x[n] \Leftrightarrow W_1(z) = W_4(z) - X(z)$$

$$w_2[n] = \alpha w_1[n] \Leftrightarrow W_2(z) = \alpha W_1(z)$$

$$w_3[n] = w_2[n] + x[n] \Leftrightarrow W_3(z) = W_2(z) + X(z)$$

$$w_4[n] = w_3[n-1] \Leftrightarrow W_4(z) = z^{-1} W_3(z)$$

$$y[n] = w_2[n] + w_4[n] \Leftrightarrow Y(z) = W_2(z) + W_4(z)$$

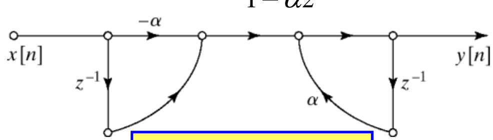


## Solve for the System Function (II)

$$W_2(z) = \frac{\alpha(z^{-1} - 1)}{1 - \alpha z^{-1}} X(z) \qquad W_4(z) = \frac{z^{-1}(1 - \alpha)}{1 - \alpha z^{-1}} X(z)$$

$$Y(z) = W_2(z) + W_4(z) = \left( \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} \right) X(z)$$

$$\Rightarrow H(z) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$



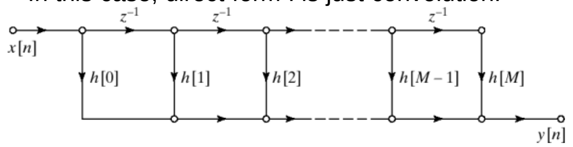

Equivalent Direct Form I

# FIR Systems

- FIR system functions have only zeros

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k}$$

- In this case, direct form I is just convolution.

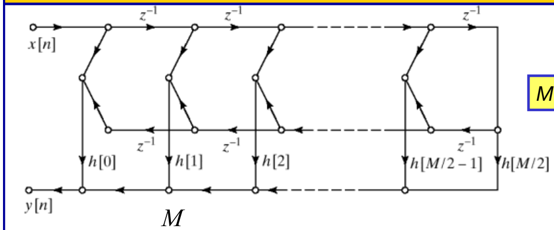



$$y[n] = (((h[0]x[n] + h[1]x[n-1]) + h[2]x[n-2]) + \ldots)$$

Multiply-accumulate (MAC)



# FIR Linear Phase Systems




*M* even

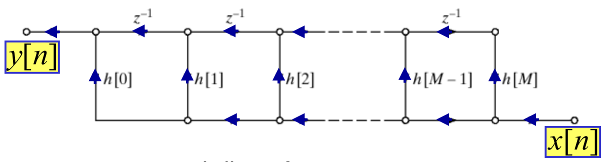
$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] \qquad h[M-n] = h[n]$$

$$y[n] = h[0](x[n] + x[M]) + h[1](x[n-1] + x[M-1]) + \ldots$$

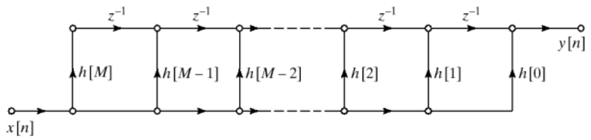


# FIR Transposed Direct Form

- FIR direct form

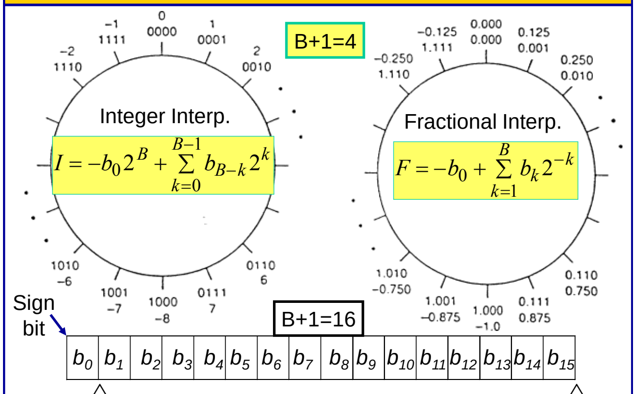



- FIR transposed direct form






# Two's Complement Numbers



## Fixed-Point Scaling - I

- 16-bit two's complement integers range in size from $-32768_{10}$ to $+32767_{10}$.
- Q notation is a convenient way for a programmer to keep track of the binary point when representing fractional numbers by integers. Consider a fractional number $a$ such that $-1 \le a < 1$, then we can represent $a$ by a Q15 integer $A$ such that $-32768 \le A \le 32767$. The relationship between $a$ and $A$ is simply

$$a = A \times 2^{-15} \quad \text{or} \quad A = a \times 2^{15}$$



## Fixed-Point Scaling - II

- Consider a fractional number $a$ such that $-1 \le a < 1$, then we can represent $a$ by a Q15 integer $A$ such that $-32768 \le A \le 32767$. The relationship between $a$ and $A$ is

$$a = A \times 2^{-15} \quad \text{or} \quad A = a \times 2^{15}$$

- For example

$$a = 0.75 \quad \Leftrightarrow \quad A = 24576_{10} \text{ Q15}$$

$$0_{\wedge}\underbrace{110000000000000}_{15 \text{ bits}}$$

Q15 means 15 bits to the right of the binary point.



## Fixed-Point Scaling - III

- Consider a mixed number $a$ such that $-4 \le a < 4$ Then we can represent $a$ by a Q13 integer $A$ such that $-32768 \le A \le 32767$. The relationship between $a$ and $A$ is

$$a = A \times 2^{-13} \quad \text{or} \quad A = a \times 2^{13}$$

- For example

$$a = 3.5 \quad \Leftrightarrow \quad A = 28672_{10} \text{ Q13}$$

$$011_{\wedge}\underbrace{1000000000000}_{13 \text{ bits}}$$



## Fixed-Point Scaling - IV

- The smallest number that can be represented by a Q15 number is

$$\Delta = \frac{\text{range}}{\text{number of possibilities}} = \frac{2}{2^{16}} = \frac{1}{2^{15}}$$

- In general, the smallest number representable as a Q*B* number will be

$$\Delta = \frac{1}{2^B}$$

- That is, in a Q*B* number, the least significant bit (LSB) has value

$$\Delta = \frac{1}{2^B}$$