ECE4270
Fundamentals of DSP
Lecture 21

Round off noise in IIR Filters and Ch7: IIR and FIR Filter Design

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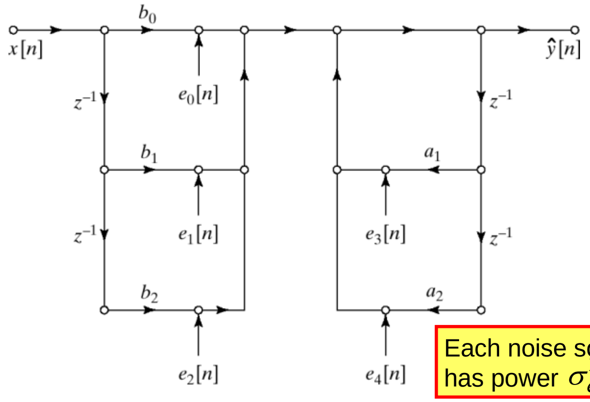
Overview of Lecture

- Roundoff noise in IIR implementations
- Example of noise analysis
- Chapter 7.
  - The filter design problem
    - FIR vs IIR filters
    - Setting up the digital filter design problem.
    - Bilinear Transform

Linear Noise Model



Each noise source has power $\sigma_e^2 = 2^{-2B}/12$

The noise sources are independent so their powers add.

Linear System with a White Noise Input

$x[n]$ → LTI System $h[n], H(e^{j\omega})$ → $y[n]$

$\phi_{xx}[m] = \sigma_x^2 \delta[m]$ $\qquad \phi_{yy}[m] = \sigma_x^2 c_{hh}[m]$

$\Phi_{xx}(e^{j\omega}) = \sigma_x^2$ $\qquad \Phi_{yy}(e^{j\omega}) = \sigma_x^2 C_{hh}(e^{j\omega})$

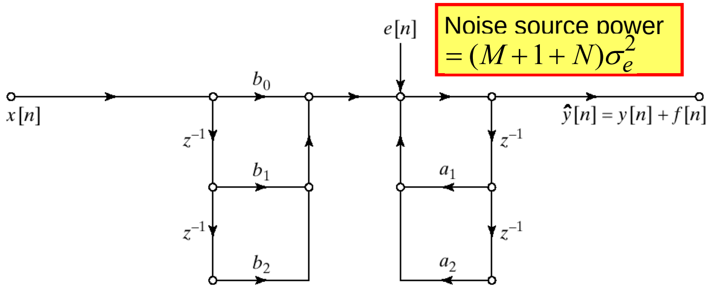
$$\phi_{yy}[m] = \phi_{xx}[m] * c_{hh}[m] = \sigma_x^2 \delta[m] * c_{hh}[m] = \sigma_x^2 c_{hh}[m]$$

$$c_{hh}[m] = \sum_{k=-\infty}^{\infty} h[m+k]h^*[k] = h[-m] * h^*[m]$$

$$\Phi_{yy}(e^{j\omega}) = \Phi_{xx}(e^{j\omega}) C_{hh}(e^{j\omega}) = \sigma_x^2 C_{hh}(e^{j\omega})$$

$$C_{hh}(e^{j\omega}) = H(e^{-j\omega}) H^*(e^{-j\omega}) = \left|H(e^{-j\omega})\right|^2$$

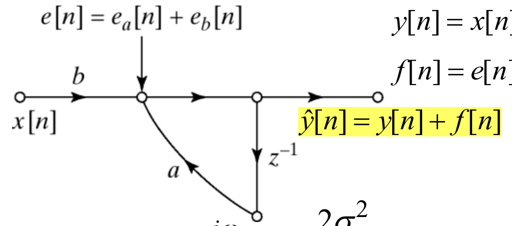
Combined Noise Sources



Noise source power $= (M+1+N)\sigma_e^2$

$$\Phi_{ff}(e^{j\omega}) = \frac{(M+1+N)\sigma_e^2}{\left|A(e^{j\omega})\right|^2}$$

Example (Output Noise Power)



$y[n] = x[n] * (ba^n u[n])$

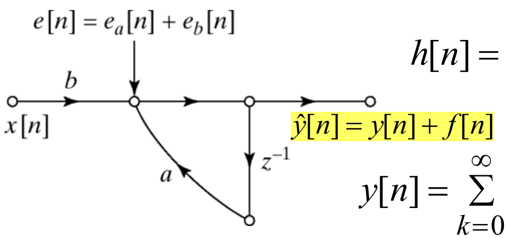
$f[n] = e[n] * (a^n u[n])$

$$\Phi_{ff}(e^{j\omega}) = \frac{2\sigma_e^2}{\left|A(e^{j\omega})\right|^2} = \frac{2\sigma_e^2}{1 + a^2 - 2a\cos\omega}$$

$$\sigma_f^2 = 2\sigma_e^2 \sum_{n=0}^{\infty} \left|h_{ef}[n]\right|^2 = 2\sigma_e^2 \sum_{n=0}^{\infty} a^{2n} = \frac{2\sigma_e^2}{1 - a^2}$$

$$\lim_{|a| \to 1} \sigma_f^2 \to \infty$$

Example (Avoiding Overflow)



$h[n] = ba^n u[n]$

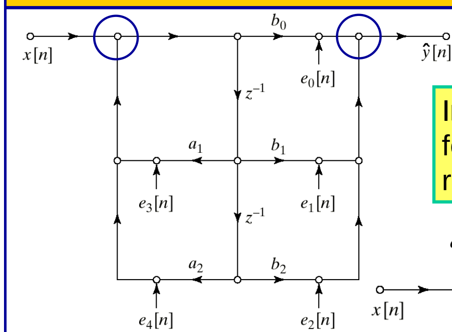
$$y[n] = \sum_{k=0}^{\infty} h[k] x[n-k]$$

$$sx_{\max} \sum_{k=0}^{\infty} |h[k]| = sx_{\max} |b| \sum_{n=0}^{\infty} |a|^n = sx_{\max} \frac{|b|}{1 - |a|} < 1$$

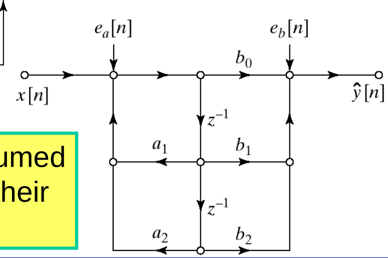
$$sx_{\max} < \frac{1 - |a|}{|b|} \to 0 \quad \text{as } |a| \to 1$$

SNR decreases as pole movies closer to unit circle.

Direct Form II Noise Modeling



Insert noise source for every word-length reduction.



Noise sources are assumed to be independent, so their powers add.

IIR Quantization Example

- Consider a 6th-order elliptic filter meeting the specifications

$$0.99 \le |H(e^{j\omega})| \le 1.01 \qquad 0 \le |\omega| \le 0.5\pi$$
$$|H(e^{j\omega})| \le 0.01 \qquad 0.56\pi \le |\omega| \le \pi$$

$$H(z) = 0.079459 \prod_{k=1}^{3} H_k(z)$$

$$= 0.079459 \prod_{k=1}^{3} \left(\frac{1 + b_{1k} z^{-1} + z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}} \right)$$

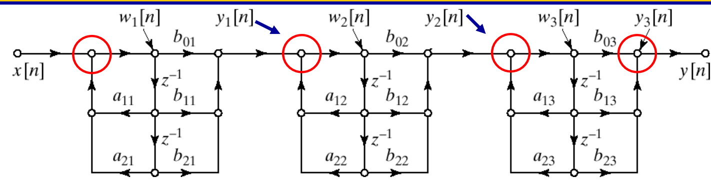
TABLE 6.4 COEFFICIENTS FOR ELLIPTIC LOWPASS FILTER IN CASCADE FORM

k	a_{1k}	a_{2k}	b_{1k}
1	0.478882	−0.172150	1.719454
2	0.137787	−0.610077	0.781109
3	−0.054779	−0.902374	0.411452

$$H(z) = \prod_{k=1}^{3} \left(\frac{b'_{0k} + b'_{1k} z^{-1} + b'_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}} \right)$$

$$b'_{01} b'_{02} b'_{03} = 0.079459$$

Absolute Scaling an IIR Digital Filter



Form II

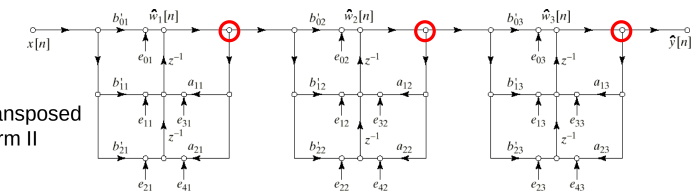
$$y_i[n] = \sum_{k=0}^{\infty} h_i[k] x[n-k]$$

$$|y_i[n]| \le \max\{x[n]\} \sum_{k=0}^{\infty} |h_i[k]| < 1 \quad \Rightarrow \quad \text{no overflow??}$$

Problem: Overflow can occur at any summing node.

Cascade Implentation (I)

Transposed Form II



Avoiding overflow:

$$H(z) = \{s_1 H_1(z)\}\{s_2 H_2(z)\}\{s_3 H_3(z)\} \qquad \text{where } s_1 s_2 s_3 = 0.079459$$

$$s_1 \max|H_1(e^{j\omega})| \le 1 \qquad s_1 s_2 \max|H_1(e^{j\omega}) H_2(e^{j\omega})| \le 1$$

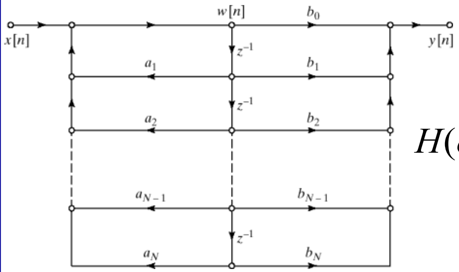
$$b'_{k1} = 0.186447 b_{k1} \qquad b'_{k2} = 0.529236 b_{k2} \qquad b'_{k3} = 0.805267 b_{k3}$$

Conclusions

- Coefficient quantization can modify the pole/zero locations and therefore the frequency response.
  - This is very severe for direct form structures
  - Quantization of coefficients may cause instability
- There is a strong interaction between roundoff noise and scaling in IIR filters
  - Roundoff noise has fixed size. Must keep signal values large, but avoid overflow.
  - This is more complicated for IIR than for FIR

IIR Filter Design

- Find B(z) and A(z) so that the filter meets the specifications on the frequency response.

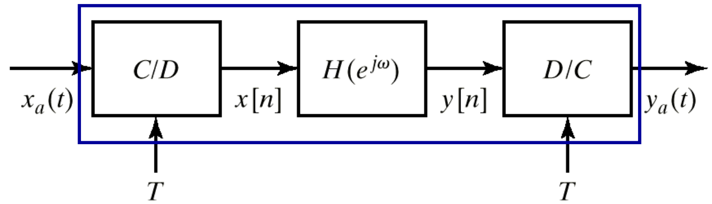


$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

Rational function approximation

Digital Filtering

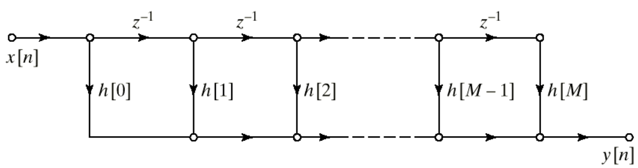


$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi / T \\ 0, & |\Omega| > \pi / T \end{cases}$$

$$H(e^{j\omega}) = H_{\text{eff}}\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

FIR Filter Design

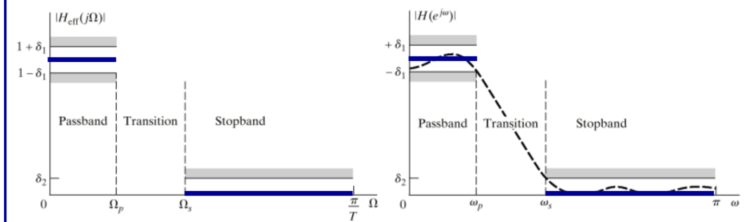
- Find the impulse response so that the filter meets a set of specifications on the frequency response.



$$H(z) = \sum_{n=0}^{M} h[n] z^{-n} \qquad H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

Polynomial approximation

Setting the Specifications



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$$H(e^{j\omega}) = H_{\text{eff}}\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

A Design Example

- The C-T specifications are ($1/T$=2000 Hz):

$$0.99 \le |H_{\text{eff}}(j\Omega)| \le 1.01, \qquad |\Omega| \le 2\pi(400)$$
$$|H_{\text{eff}}(j\Omega)| \le 0.001, \quad 2\pi(600) \le |\Omega| \le 2\pi(1000)$$
$$\text{i.e., } \Omega_p = 2\pi(400) \text{ and } \Omega_s = 2\pi(600).$$

- The corresponding D-T specifications are:

$$0.99 \le |H(e^{j\omega})| \le 1.01, \qquad |\omega| \le 0.4\pi$$
$$|H(e^{j\omega})| \le 0.001, \quad 0.6\pi \le |\omega| \le \pi$$
$$\text{i.e., } \omega_p = \Omega_p T = 0.4\pi \text{ and } \omega_s = \Omega_s T = 0.6\pi.$$

Bilinear Transformation

- We simply transform an analog filter $H_c(s)$ into a digital filter $H(z)$ with the complex mapping

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right); \text{ i.e., } H(z) = H_c(s)\Big|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}.$$

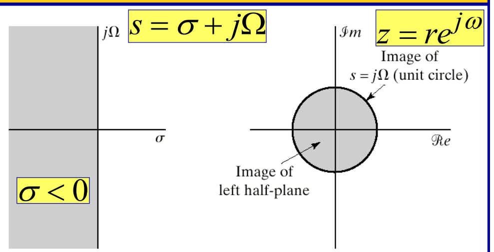
$$z = \left(\frac{1+sT_d/2}{1-sT_d/2}\right) \Rightarrow re^{j\omega} = \left(\frac{1+\sigma T_d/2 + j\Omega T_d/2}{1-\sigma T_d/2 - j\Omega T_d/2}\right)$$

$$\sigma = 0 \Rightarrow r = 1 \text{ and } \frac{\omega}{2} = \arctan(\Omega T_d/2)$$
$$\sigma < 0 \Rightarrow r < 1 \text{ and } \sigma > 0 \Rightarrow r > 1$$

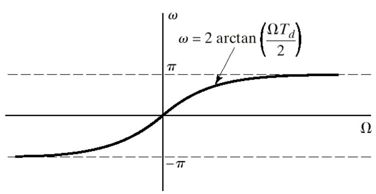
Bilinear Transformation

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$z = \left(\frac{1+sT_d/2}{1-sT_d/2}\right)$$



$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$



Bilinear Transformation of $H_c(j\Omega)$

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$

$$\frac{\omega}{2} = \arctan(\Omega T_d/2)$$

$$\Omega = \frac{2}{T_d}\tan(\omega/2)$$

$$H(e^{j\omega}) = H_c\left(j\frac{2}{T_d}\tan(\omega/2)\right)$$