ECE4270
Fundamentals of DSP

Lecture 22

IIR Filter Design

School of ECE
Center for Signal and Information Processing
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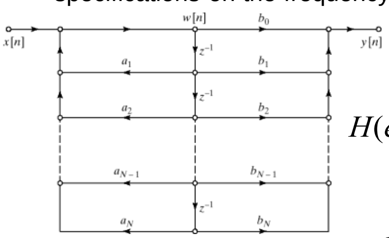
Overview of Lecture

- Chapter 7.
  - The filter design problem
    - FIR vs IIR filters
    - Setting up the digital filter design problem.
    - Bilinear Transform
- Design of IIR filters by bilinear transformation.
  - Mapping of s-plane into z-plane
  - Frequency warping
  - Example (Butterworth filter)
  - Chebyshev and Elliptic IIR filters

IIR Filter Design

- Find B(z) and A(z) so that the filter meets the specifications on the frequency response.



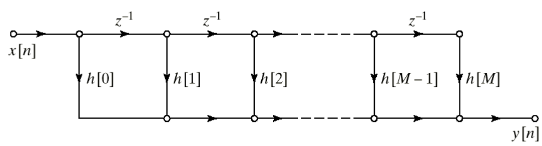
$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

Rational function approximation

FIR Filter Design

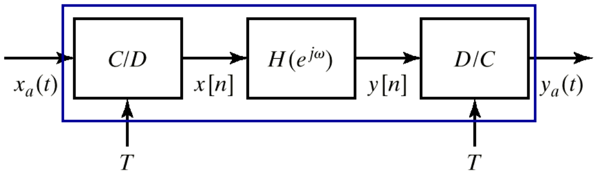
- Find the impulse response so that the filter meets a set of specifications on the frequency response.



$$H(z) = \sum_{n=0}^{M} h[n] z^{-n} \qquad H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

Polynomial approximation

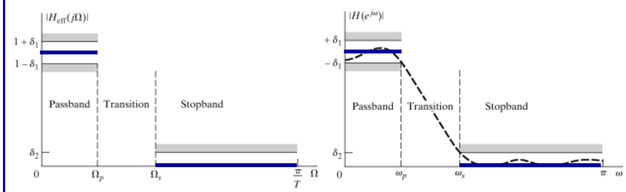
Digital Filtering



$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi / T \\ 0, & |\Omega| > \pi / T \end{cases}$$

$$H(e^{j\omega}) = H_{\text{eff}}\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

Setting the Specifications



$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi / T \\ 0, & |\Omega| > \pi / T \end{cases}$$

$$H(e^{j\omega}) = H_{\text{eff}}\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

A Design Example

- The C-T specifications are (*1/T*=2000 Hz):

$$0.99 \le |H_{\text{eff}}(j\Omega)| \le 1.01, \qquad |\Omega| \le 2\pi(400)$$
$$|H_{\text{eff}}(j\Omega)| \le 0.001, \quad 2\pi(600) \le |\Omega| \le 2\pi(1000)$$
$$\text{i.e., } \Omega_p = 2\pi(400) \text{ and } \Omega_s = 2\pi(600).$$

- The corresponding D-T specifications are:

$$0.99 \le |H(e^{j\omega})| \le 1.01, \qquad |\omega| \le 0.4\pi$$
$$|H(e^{j\omega})| \le 0.001, \quad 0.6\pi \le |\omega| \le \pi$$
$$\text{i.e., } \omega_p = \Omega_p T = 0.4\pi \text{ and } \omega_s = \Omega_s T = 0.6\pi.$$

Bilinear Transformation

- We simply transform an analog filter $H_c(s)$ into a digital filter $H(z)$ with the complex mapping

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right); \text{ i.e., } H(z) = H_c(s)\Big|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}.$$

$$z = \left(\frac{1+sT_d/2}{1-sT_d/2}\right) \Rightarrow re^{j\omega} = \left(\frac{1+\sigma T_d/2 + j\Omega T_d/2}{1-\sigma T_d/2 - j\Omega T_d/2}\right)$$

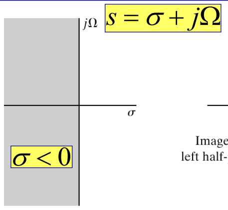
$$\sigma = 0 \Rightarrow r = 1 \text{ and } \frac{\omega}{2} = \arctan(\Omega T_d/2)$$

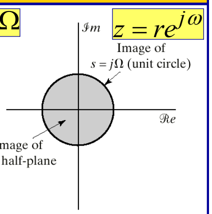
$$\sigma < 0 \Rightarrow r < 1 \text{ and } \sigma > 0 \Rightarrow r > 1$$

Bilinear Transformation

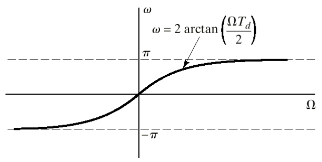
$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$z = \left(\frac{1+sT_d/2}{1-sT_d/2}\right)$$





$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$



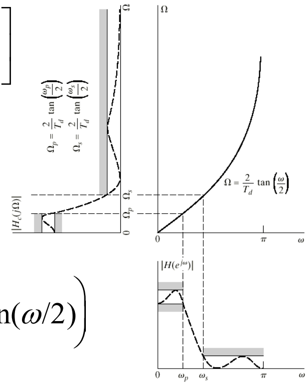
Bilinear Transformation of $H_c(j\Omega)$

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$

$$\frac{\omega}{2} = \arctan(\Omega T_d/2)$$

$$\Omega = \frac{2}{T_d}\tan(\omega/2)$$

$$H(e^{j\omega}) = H_c\left(j\frac{2}{T_d}\tan(\omega/2)\right)$$



Simple (Butterworth) Example

$$H_c(s) = \frac{1}{s^2 + \sqrt{2}s + 1} \Rightarrow |H_c(j\Omega)|^2 = \frac{1}{1+\Omega^4}$$

- The digital filter has system function

$$H(z) = \frac{1}{\left(\frac{2}{T_d}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \sqrt{2}\left(\frac{2}{T_d}\frac{1-z^{-1}}{1+z^{-1}}\right) + 1}$$

$$\left|H(e^{j\omega})\right|^2 = \left|H_c\left(j\frac{2}{T_d}\tan(\omega/2)\right)\right|^2 = \frac{1}{1+\left(\frac{2}{T_d}\tan(\omega/2)\right)^4}$$

A Design Example (I)

- The D-T specifications are:

$$0.99 \le |H(e^{j\omega})| \le 1.01, \qquad |\omega| \le 0.4\pi$$
$$|H(e^{j\omega})| \le 0.001, \quad 0.6\pi \le |\omega| \le \pi$$

i.e., $\omega_p = \Omega_p T = 0.4\pi$ and $\omega_s = \Omega_s T = 0.6\pi$.

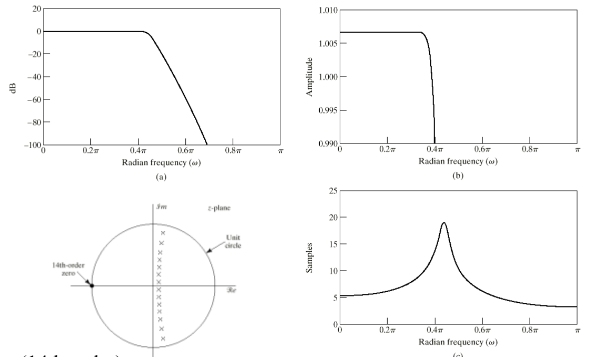
- The continuous-time prototype filter $H_c(j\Omega)$ must satisfy:

$$0.99 \le |H_c(j\Omega)| \le 1.01, \qquad |\Omega| \le \frac{2}{T_d}\tan(0.4\pi/2)$$
$$|H_c(j\Omega)| \le 0.001, \quad \frac{2}{T_d}\tan(0.6\pi/2) \le |\Omega| < \infty$$

Butterworth Approx. in MATLAB (II)

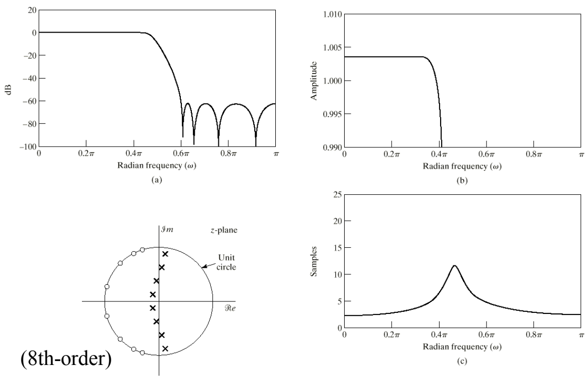
```
»[N,wp]=buttord(.4,.6,-20*log10(.99),-20*log10(.001))
N = 14
wp = 0.44490626110897
» [b,a]=butter(N,wp)
b =
  Columns 1 through 7
    0.0001    0.0011    0.0071    0.0284    0.0782    0.1563    0.2345
  Columns 8 through 14
    0.2680    0.2345    0.1563    0.0782    0.0284    0.0071    0.0011
  Column 15
    0.0001
a =
  Columns 1 through 7
    1.0000   -1.5395    2.9473   -2.8363    2.7428   -1.7703    1.0616
  Columns 8 through 14
   -0.4647    0.1794   -0.0516    0.0123   -0.0021    0.0003   -0.0000
  Column 15
    0.0000
```


Butterworth Approximation (III)



(14th-order)

Chebyshev Approximation (IV)



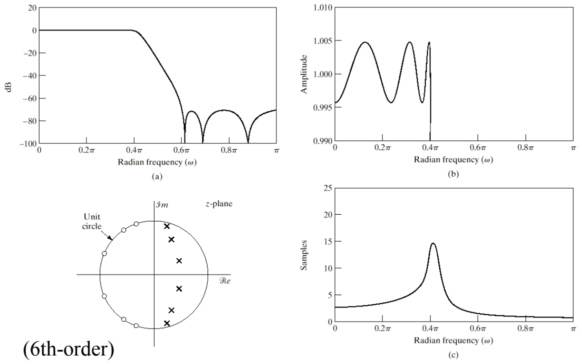
(8th-order)

Elliptic Approximation in MATLAB (V)

```
»[N,wp]=ellipord(.4,.6,-20*log10(.99),-20*log10(.001))
N =
     6
wp =
    0.4000

»[b,a]=ellip(N,-20*log10(.99),-20*log10(.001),wp)
b =
    0.0208    0.0590    0.1068    0.1265    0.1068    0.0590    0.0208
a =
    1.0000   -1.9585    2.8916   -2.5155    1.5627   -0.5906    0.1148
```

Elliptic Approximation (VI)



(6th-order)