ECE4270
Fundamentals of DSP

Lecture 24

Optimal FIR Filter Design & DFT (Chapter 8)

School of ECE
Center for Signal and Information Processing
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Overview of Lecture

- Design of FIR filters by the window method (Last Lecture)
- Design of differentiator
- Multiband design
- Optimum Filter Design
  - The Parks-McClellan algorithm

- Chapter 8
- Discrete Fourier Transform (DFT)
- The DFT as a Sampled DTFT

Kaiser Window Design Method

$$w[n] = \begin{cases} \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha = M/2$$

$$\Delta\omega = \omega_s - \omega_p \quad \text{and} \quad A = -20\log_{10}\delta$$

$$M = \frac{A-8}{2.285\Delta\omega} \quad \Rightarrow \quad \text{required to meet specs}$$

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 < A < 50 \\ 0.0 & A < 21 \end{cases}$$

Digital "Differentiator"

- Suppose that we want to design an FIR filter such that

$$H_{\text{eff}}(j\Omega) = j\Omega, \quad |\Omega| < \frac{\pi}{T}$$

- The required digital filter must approximate

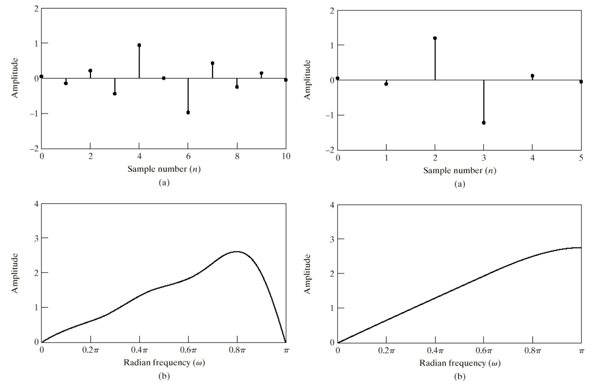
$$H(e^{j\omega}) = \frac{j\omega}{T} e^{-j\omega M/2}, \quad |\omega| < \pi$$

- The desired impulse response is

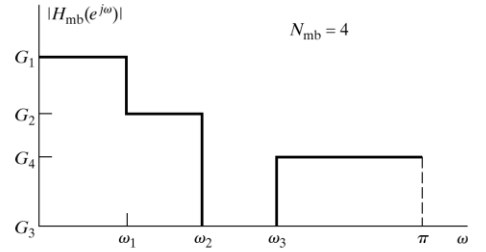
$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{j\omega}{T}\right) e^{-j\omega M/2} e^{j\omega n} d\omega$$

$$h_d[n] = \frac{\cos[\pi(n-M/2)]}{(n-M/2)T} - \frac{\sin[\pi(n-M/2)]}{\pi(n-M/2)^2 T}, \quad -\infty < n < \infty$$

Kaiser Window Differentiators



General Frequency Selective Filter



$$h_{mb}[n] = \sum_{k=1}^{N_{mb}} (G_k - G_{k+1}) \frac{\sin \omega_k (n - M/2)}{\pi (n - M/2)}$$

Parks and McClellan, 1972

Chebyshev Approximation for Nonrecursive Digital Filters with Linear Phase

THOMAS W. PARKS, MEMBER, IEEE, AND JAMES H. McCLELLAN, STUDENT MEMBER, IEEE

Abstract—An efficient procedure for the design of finite-length impulse response filters with linear phase is presented. The algorithm obtains the optimum Chebyshev approximation on separate intervals corresponding to passbands and/or stopbands, and is capable of designing very long filters. This approach allows the exact specification of arbitrary band-edge frequencies as opposed to previous algorithms which could not directly control pass- and stopband locations and could only obtain $(N-1)/2$ different band-edge locations for a length N low-pass filter, for fixed δ_1 and δ_2.

As an aid in practical application of the algorithm, several graphs are included to show relations among the parameters of filter length, transition width, band-edge frequencies, passband ripple, and stopband attenuation.

capable of designing longer filters. The algorithms in [7], [8] result in exactly the same filter and will be called an extraripple design in this paper.

The detailed description of the new procedure described here is in terms of low-pass filters. Modifications for the general bandpass case are included. Linear-phase digital filters of length $2n+1$ have a transfer function

$$G(Z) = \sum_{k=0}^{2n} h_k Z^{-k} \quad (1)$$

T. W. Parks and J. H. McClellan, *IEEE Trans. Circuit Theory*, CT-19, pp. 189-194, March, 1972.

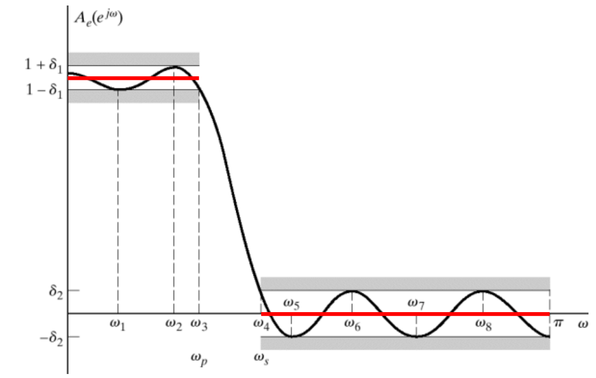
The Parks McClellan Algorithm

- Uses the Remez exchange algorithm to iteratively find the impulse response that minimizes the maximum approximation error over a set of closed intervals in the frequency-domain.

$$E(\omega) = W(\omega)[H_d(e^{j\omega}) - H(e^{j\omega})]$$

- Leads to equiripple approximations that are optimum in sense of smallest approximation error for a given transition width.

Optimum FIR Filter



Linear Phase Type I FIR Filter

- Zero-phase impulse response:

$$h_e[-n] = h_e[n] \quad -L \le n \le L$$

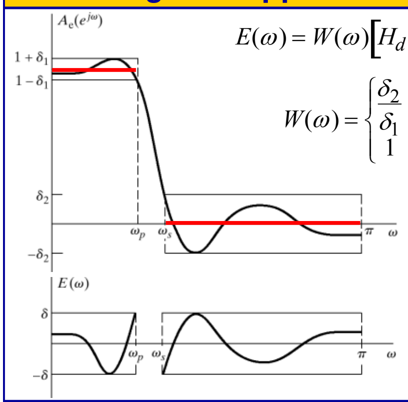
- Frequency response:

$$A_e(e^{j\omega}) = \sum_{n=-L}^{L} h_e[n]e^{-j\omega n}$$
$$= h_e[0] + \sum_{n=1}^{L} \left(h_e[n]e^{-j\omega n} + h_e[-n]e^{j\omega n}\right)$$
$$= h_e[0] + \sum_{n=1}^{L} 2h_e[n]\cos\omega n = \sum_{k=0}^{L} a_k(\cos\omega)^k$$

- Causal version:

$$h[n] = h_e[n-L] \quad \Leftrightarrow \quad H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega L}$$

Weighted Approximation Error



$$E(\omega) = W(\omega)\left[H_d(e^{j\omega}) - A_e(e^{j\omega})\right]$$

$$W(\omega) = \begin{cases} \frac{\delta_2}{\delta_1} & 0 \le \omega \le \omega_p \\ 1 & \omega_s \le \omega \le \pi \end{cases}$$

The Alternation Theorem

- Weighted approximation error:

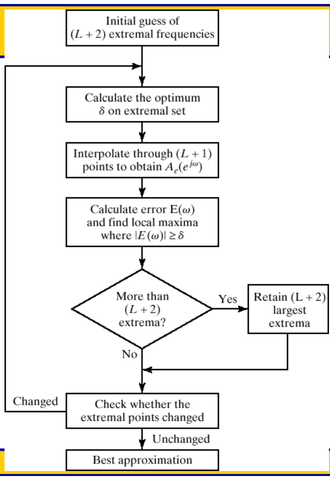
$$E(\omega) = W(\omega)\left[H_d(e^{j\omega}) - A_e(e^{j\omega})\right]$$

- Minimize the maximum error over a set of frequencies:

$$F = \{\omega : 0 \le \omega \le \omega_p \text{ and } \omega_s \le \omega \le \pi\}$$

- The optimum approximation alternates between +δ and -δ at least *L*+2 times in *F*. The maximum number of alternations is *L*+3.

$$\|E\| = \max_{\omega \in F}\left[|E(\omega)|\right] \qquad \delta = \min_{h_e[n]}\{\|E\|\}$$



Design Formula

- Kaiser obtained the following design formula by curve fitting many examples:

$$M = \frac{-10\log_{10}(\delta_1\delta_2)-13}{2.324\Delta\omega}$$

- m=(-10*log10(.01*.001)-13) / (2.324*(0.6-0.4)*pi) =25.3388

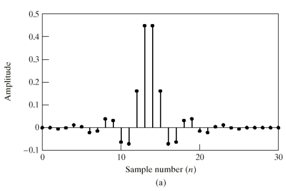
- MATLAB example:

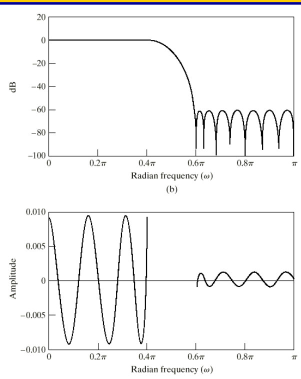
» [M,Fo,Mo,W] = remezord([.4,.6], [1 0], [0.01 0.001], 2);

» [h,delta]=remez(M,Fo,Mo,W);

Parks McClellan Lowpass Design

$M = 27$





Comparison of Filter Structures

- Complexity is proportional to amount of computation and storage plus program storage and computational cycles.
- FIR direct form - (M+1) coefficients
  - (M+1) multiplications, M additions
  - (M+1) coefficients, M delays (registers)
- IIR cascade form - N_s second-order sections.
  - 5N_s multiplications, 5N_s additions
  - 5N_s coefficients, 5N_s delays

Lowpass Filter Implementations

- Specifications of lowpass filter $1/T = 2000$ Hz
  $$0.99 \le |H(e^{j\Omega T})| \le 1.01 \qquad 0 \le |\Omega| \le 2\pi(400)$$
  $$|H(e^{j\Omega T})| \le 0.01 \qquad 2\pi(600) \le |\Omega| \le 2\pi(1000)$$
- These specs met by the following approx.
  - Butterworth - 12th-order
  - Chebyshev - 8th-order
  - Elliptic - 6th-order
  - Kaiser window - 37 sample impulse response
  - Parks-McClellan - 27 sample impulse response

Comparison of Lowpass Filters

Approx. Method	Order M or N	Total Mults.	Total Adds	Total Storage	TMS320 Cycles
Butter	14	35	28	49	109
Cheby	8	20	16	28	64
elliptic	6	18	12	21	49
Kaiser	37	38	37	74	52
P-Mc	27	28	27	54	42

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The Discrete Fourier Transform (DFT)

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Review of the DTFT

- Definition: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(z)\big|_{z=e^{j\omega}}$
- Inverse transform: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
- Periodicity: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$
- Convolution theorems:

$$y[n]=x[n]*h[n] \Leftrightarrow Y(e^{j\omega})=X(e^{j\omega})\cdot H(e^{j\omega})$$

$$y[n]=w[n]\cdot x[n] \Leftrightarrow Y(e^{j\omega})=\frac{1}{2\pi}W(e^{j\omega})*X(e^{j\omega})$$

The Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1,\ldots,N-1$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn} \qquad n = 0,1,\ldots,N-1$$

where $W_N = e^{-j(2\pi/N)}$.

- Exact representation of finite-length or periodic sequences ($x[n+N]=x[n]$).
- $X[k]$ and $x[n]$ can be computed efficiently by the FFT. (Gauss knew about it, Cooley and Tukey rediscovered it at just the right time.)

A Simple (but important) Example

- Let P[k]=1, for k=0,1,2,…,N-1. Then

$$p[n] = \frac{1}{N}\sum_{k=0}^{N-1} e^{j(2\pi/N)kn} = \frac{1}{N}\frac{1-e^{j(2\pi/N)nN}}{1-e^{j(2\pi/N)n}}$$

$$p[n] = \begin{cases} 1 & n=0,\pm N,\pm 2N,\ldots \\ 0 & \text{otherwise} \end{cases} = \sum_{r=-\infty}^{\infty} \delta[n+rN]$$

- DFTs (and inverse DFTs) are inherently periodic with period N.

$$p[n+N] = \frac{1}{N}\sum_{k=0}^{N-1} P[k] e^{j(2\pi/N)k(n+\not{N})} = p[n]$$

The DFT as a Sampled DTFT

- The DTFT of an N-point sequence is

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

- Sample the DTFT at $\omega_k = (2\pi/N)k,\ k = 0,1,\ldots,N-1.$
- The result is identical to the DFT

$$X(e^{j\omega})\Big|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} = X[k]$$

- If we compute the inverse DFT, we obtain

$$\tilde{x}[n] = \frac{1}{N}\sum_{n=0}^{N-1} X(e^{j(2\pi/N)k})\, e^{j(2\pi/N)kn} = \sum_{r=-\infty}^{\infty} x[n+rN]$$