ECE4270 Fundamentals of DSP

Lecture 25

The Discrete Fourier Transform (DFT)

School of ECE
Center for Signal and Information Processing
Georgia Institute of Technology

Overview of Lecture

- Chapter 8
- Discrete Fourier Transform (DFT)
- The DFT as a Sampled DTFT
- Circular (cyclic) Convolution
- Linear Convolution via DFT

Review of the DTFT

- Definition: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(z)\big|_{z=e^{j\omega}}$
- Inverse transform: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
- Periodicity: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$
- Convolution theorems:

$$y[n]=x[n]*h[n] \Leftrightarrow Y(e^{j\omega})=X(e^{j\omega})\cdot H(e^{j\omega})$$

$$y[n]=w[n]\cdot x[n] \Leftrightarrow Y(e^{j\omega})=\frac{1}{2\pi}W(e^{j\omega})*X(e^{j\omega})$$

The Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} \qquad k = 0,1,\ldots,N-1$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn} \qquad n = 0,1,\ldots,N-1$$

where $W_N = e^{-j(2\pi/N)}$.

- Exact representation of finite-length or periodic sequences ($x[n+N]=x[n]$).
- $X[k]$ and $x[n]$ can be computed efficiently by the FFT. (Gauss knew about it, Cooley and Tukey rediscovered it at just the right time.)

A Simple (but important) Example

- Let P[k]=1, for k=0,1,2,…,N-1. Then

$$p[n] = \frac{1}{N}\sum_{k=0}^{N-1} e^{j(2\pi/N)kn} = \frac{1}{N}\frac{1-e^{j(2\pi/N)nN}}{1-e^{j(2\pi/N)n}}$$

$$p[n] = \begin{cases} 1 & n=0,\pm N,\pm 2N,\ldots \\ 0 & \text{otherwise} \end{cases} = \sum_{r=-\infty}^{\infty} \delta[n+rN]$$

- DFTs (and inverse DFTs) are inherently periodic with period N.

$$p[n+N] = \frac{1}{N}\sum_{k=0}^{N-1} P[k]e^{j(2\pi/N)k(n+\cancel{N})} = p[n]$$

The DFT as a Sampled DTFT

- The DTFT of an *N*-point sequence is

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

- Sample the DTFT at $\omega_k = (2\pi/N)k,\ k = 0,1,\ldots,N-1.$
- The result is identical to the DFT

$$X(e^{j\omega})\Big|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn} = X[k]$$

- If we compute the inverse DFT, we obtain

$$\tilde{x}[n] = \frac{1}{N}\sum_{n=0}^{N-1} X(e^{j(2\pi/N)k})\, e^{j(2\pi/N)kn} = \sum_{r=-\infty}^{\infty} x[n+rN]$$

Proof of DFT Sampling Theorem

- Consider a signal with DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Sample it to get a DFT

$$X[k] = X(e^{j(2\pi/N)k}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(2\pi/N)kn}$$

- Compute inverse DFT

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} x[m]e^{-j(2\pi/N)km}e^{j(2\pi/N)kn}$$

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\frac{1}{N}\sum_{k=0}^{N-1} e^{j(2\pi/N)k(n-m)} = \sum_{r=-\infty}^{\infty} x[n+rN]$$

DFT Sampling Theorem

- If we sample the DTFT of *x*[n] at N equally spaced frequencies, the corresponding periodic sequence (through the inverse DFT) is the time-domain aliased sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$$

- Then if *x*[n]=0 for *n*<0 and for *n*>*N-1*, the copies of x[n] do not overlap so we can write

$$\tilde{x}[n] = x[((n))_N] = x[n \text{ modulo } N]$$

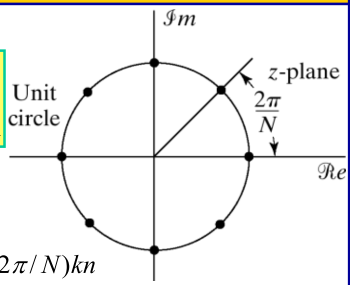
- Therefore:

$$x[n] = \begin{cases} \tilde{x}[n] & n = 0,1,\ldots,N-1 \\ 0 & \text{otherwise} \end{cases}$$

Sampling the *z*-Transform

$z = e^{j(2\pi/N)kn},$

$k = 0,1,2,\ldots,N-1$



$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}$$

$$= X(z)\big|_{z=e^{j(2\pi/N)kn}}, \quad k = 0,1,2,\ldots,N-1$$

Example of Sampling the DTFT

- Input sequence: $x[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$
- DTFT

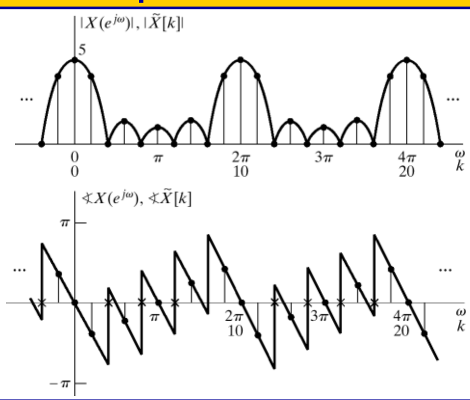
$$X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} = \frac{1-e^{-j\omega 5}}{1-e^{-j\omega}} = \frac{\sin(5\omega/2)}{\sin(\omega/2)}e^{-j\omega 2}$$

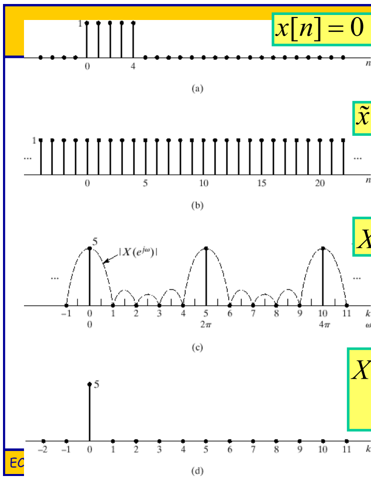
- Sampled DTFT is the DFT

$$X(e^{j(2\pi/N)k}) = \sum_{n=0}^{4} e^{-j(2\pi/N)kn} = X[k]$$

$$= \frac{\sin(5(2\pi/N)k/2)}{\sin((2\pi/N)k/2)}e^{-j(2\pi/N)k2}$$

Sampled DTDF is DFT





$x[n] = 0 \quad n < 0 \text{ and } n > M$

$M = 4$

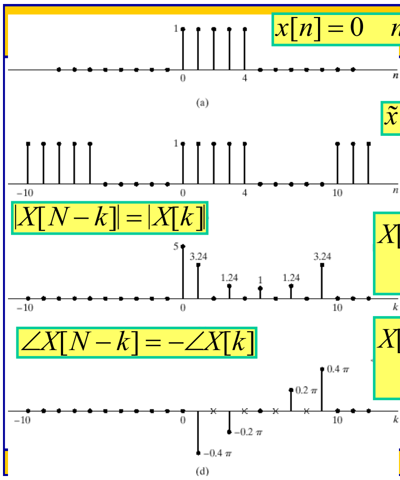
$\tilde{x}[n] = x[((n))_N]$

$N = 5$

$X[k] = X(e^{j(2\pi/N)k})$

$X[k] = X(e^{j(2\pi/N)k}),$

$k = 0,1,\ldots,N-1$



$x[n]=0 \quad n<0$ and $n>M$

$M=4$

$\tilde{x}[n]=x[((n))_N]$

$N=10$

$|X[N-k]|=|X[k]|$

$X[k]=X(e^{j(2\pi/N)k}),\quad k=0,1,\ldots,N-1$

$\angle X[N-k]=-\angle X[k]$

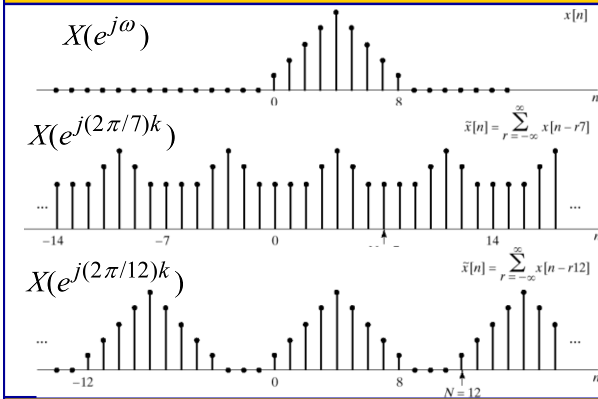
$X[N-k]=X^*[k],\quad k=0,1,\ldots,N-1$

Time-Domain Aliasing

$X(e^{j\omega})$

$X(e^{j(2\pi/7)k})$

$X(e^{j(2\pi/12)k})$



Basic Properties of the DFT

- If the DFT is evaluated outside of $0\le n\le N-1$, it repeats periodically as $\tilde{x}[n]=x[((n))_N]$.

$$\tilde{x}[n+N]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-k(n+N)}=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn}=\tilde{x}[n]$$

- Circular shift

$$x_1[n]=x[((n-m))_N] \quad\Leftrightarrow\quad X_1[k]=W_N^{km}X[k]$$

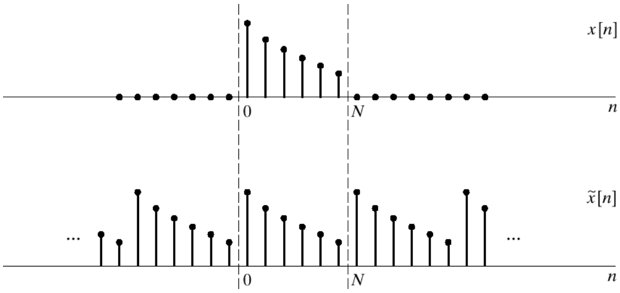
- Circular convolution

$$y[n]=\sum_{m=0}^{N-1}x[m]h[((n-m))_N] \quad\Leftrightarrow\quad Y[k]=X[k]H[k]$$

DFT Theorems and Properties

	Finite-Length Sequence (Length N)	N-point DFT (Length N)
1.	$x[n]$	$X[k]$
2.	$x_1[n],\ x_2[n]$	$X_1[k],\ X_2[k]$
3.	$ax_1[n]+bx_2[n]$	$aX_1[k]+bX_2[k]$
4.	$X[n]$	$Nx[((-k))_N]$
5.	$x[((n-m))_N]$	$W_N^{km}X[k]$
6.	$W_N^{-\ell n}x[n]$	$X[((k-\ell))_N]$
7.	$\sum_{m=0}^{N-1}x_1(m)x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8.	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1}X_1(\ell)X_2[((k-\ell))_N]$

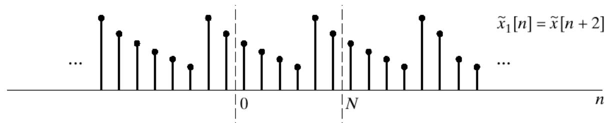
Circular Shift - I



$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN] = x[((n))_N]$$

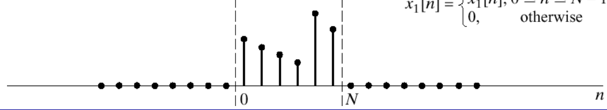
Circular Shift - II

$$\tilde{x}_1[n] = \tilde{x}[n+2] = \sum_{r=-\infty}^{\infty} x[n + 2 + rN]$$

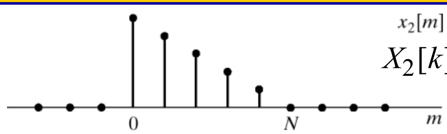


$$x_1[n] = x[((n+2))_N]$$

$$x_1[n] = \begin{cases} \tilde{x}_1[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$



Circular Convolution - I



$$X_2[k]$$

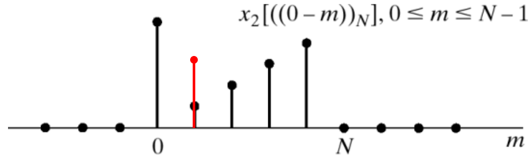
$$X_1[k] = e^{-j(2\pi/N)k}$$

$$X_3[k] = X_1[k]X_2[k] = e^{-j(2\pi/N)k} X_2[k]$$

$$x_3[n] = \sum_{m=0}^{N-1} \delta[m-1] x_2[((n-m))_N] = x_2[((n-1))_N]$$

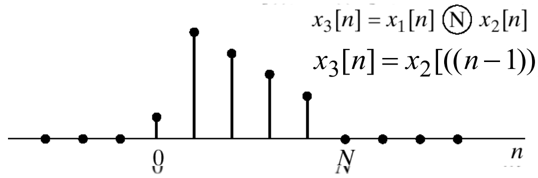
Circular Flipping and Shifting

$$x_2[((0-m))_N], 0 \le m \le N-1$$

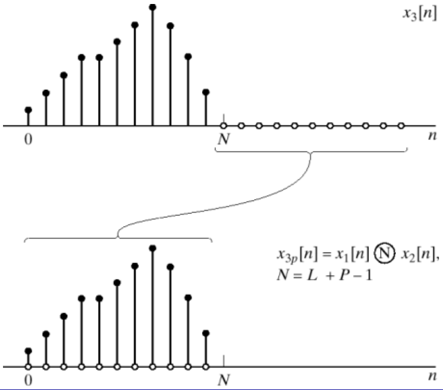


$$x_3[n] = x_1[n] \; \textcircled{N} \; x_2[n]$$

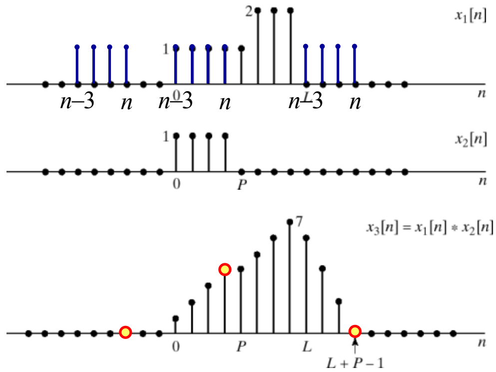
$$x_3[n] = x_2[((n-1))_N]$$



Aliased Convolution - III



"Linear Convolution"



Analysis for Linear Convolution

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m] \Leftrightarrow X_3(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$$

$$X_3(e^{j(2\pi/N)k}) = X_1(e^{j(2\pi/N)k})X_2(e^{j(2\pi/N)k})$$

$$\Rightarrow X_3[k] = X_1[k]X_2[k]$$

$$x_{3p}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]X_2[k]e^{j(2\pi/N)kn}$$

$$x_{3p}[n] = \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

$$x_{3p}[n] = \sum_{r=-\infty}^{\infty} x_3[n-rN] \quad n = 0,1,\ldots,N-1$$

To avoid overlap, pick $N > L+P-2$